Journal of Computational Physics 229 (2010) 4259–4278

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Splitting multisymplectic integrators for Maxwell's equations

Linghua Kong [a,*], Jialin Hong [b], Jingjing Zhang [b]

[a] School of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
[b] State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, AMSS, CAS, P.O. Box 2719, Beijing 100190, China

ARTICLE INFO

Article history:
Received 1 May 2009
Received in revised form 19 January 2010
Accepted 11 February 2010
Available online 19 February 2010

MSC:
65M06
65M12
65Z05
70H15

Keywords:
Maxwell's equation
Local one-dimensional method
Multisymplectic integrator
Runge–Kutta method
Conservation law

ABSTRACT

In the paper, we describe a novel kind of multisymplectic method for three-dimensional (3-D) Maxwell's equations. Splitting the 3-D Maxwell's equations into three local one-dimensional (LOD) equations, then applying a pair of symplectic Runge–Kutta methods to discretize each resulting LOD equation, it leads to splitting multisymplectic integrators. We say this kind of schemes to be LOD multisymplectic scheme (LOD-MS). The discrete conservation laws, convergence, dispersive relation, dissipation and stability are investigated for the schemes. Theoretical analysis shows that the schemes are unconditionally stable, non-dissipative, and of first order accuracy in time and second order accuracy in space. As a reduction, we also consider the application of LOD-MS to 2-D Maxwell's equations. Numerical experiments match the theoretical results well. They illustrate that LOD-MS is not only efficient and simple in coding, but also has almost all the nature of multisymplectic integrators.

1. Introduction

It has been widely recognized that the symplectic structure-preserving numerical methods have the remarkable superiority to conventional numerical methods when applied to Hamiltonian ODEs and PDEs [1–3], such as, long-term behavior, symplectic structure-preserving, etc. At the end of last century, symplectic integrators have been generalized to multisymplectic ones [4,5]. The multisymplectic Hamiltonian partial differential equations (HPDEs) with m spatial dimensions read [5]

$$M\mathbf{z}_t + \sum_{k=1}^{m} K_k \mathbf{z}_{x_k} = \nabla_{\mathbf{z}} S(\mathbf{z}), \tag{1.1}$$

where M and $K_k,\ k = 1, 2, \ldots, m$, are skew-symmetric matrices and mean symplectic structures, and $S(\mathbf{z})$ is a smooth function which is called the Hamiltonian function or the energy functional. It is well-known that along the solutions $\mathbf{z}(x_1, x_2, \ldots, x_m, t)$ of an HPDE, it gives rise to the multisymplectic conservation law (MSCL)

$$\frac{\partial}{\partial t}\omega + \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\kappa_k = 0, \tag{1.2}$$

* Corresponding author.
E-mail addresses: konglh@mail.ustc.edu.cn (L. Kong), hjl@lsec.cc.ac.cn (J. Hong), zhangjj@lsec.cc.ac.cn (J. Zhang).

0021-9991/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2010.02.010

with the differential 2-forms $\omega = d\mathbf{z} \wedge M d\mathbf{z}$, $\kappa_k = d\mathbf{z} \wedge K_k d\mathbf{z}$, $k = 1, 2, \ldots, m$, where dz fulfills the variational equation of multisymplectic system (1.1).

Furthermore, for the general multisymplectic system (1.1), there is a local energy conservation law which is in geometric form

$$\frac{\partial}{\partial t} P(\mathbf{z}) + \sum_{i=1}^{m} \frac{\partial}{\partial x_k} Q_k(\mathbf{z}) = 0, \tag{1.3}$$

provided that $S(\mathbf{z})$ is independent of the variables x, t explicitly, where

$$P(\mathbf{z}) = S(\mathbf{z}) - \frac{1}{2} \sum_{k=1}^{m} \mathbf{z}^T K_k \mathbf{z}_{x_k}, \quad Q_k(\mathbf{z}) = \frac{1}{2} \mathbf{z}^T K_k \mathbf{z}_t, \quad k = 1, 2, \ldots, m.$$

Numerical methods which preserve the multisymplectic geometric structure are desirable because of their inviting advantages, for instance, the stability of long-term numerical simulation, the preservation of local conservation laws, and so on. Many scientists are injecting themselves to the field, and rapid progress has been made during the last 10 years for various HPDEs (see [4–14] and references therein). The most important and popular class of multisymplectic methods, concatenating of symplectic Runge–Kutta (SRK) methods or symplectic partitioned Runge–Kutta (SPRK) methods, are completely implicit, especially for nonseparable Hamiltonian system. It is straightforward to apply these methods to multi-dimensional HPDEs theoretically. However, it is difficult in programming because of the huge scale of the algebraic equations and substantial computational cost. For example, for a 3-D problem, it is required to solve at least one 10^6 scale algebraic equation at every time step provided that the considered spatial domain is divided into $100 \times 100 \times 100$ cells. This is insolvable by a personal computer (PC) because of the limitation of memory and the performance of CPU up to now.

The alternating direction implicit (ADI) method, the local one-dimensional (LOD) method and the fractional step (FS) method are originally devised to solve multi-dimensional parabolic problems by Peaceman, Douglas and Rachford [18–21]. The most magnetic and popular merits of these methods are economic in the use of memory and CPU time. For instance, for the previous example, we only need to solve about thirty thousands 100 scale algebraic systems at very time step if we adopt the LOD strategy. There is no more difficulties for the problem from the aspect of memory and CPU.

To get over the difficulties with respect to memory and computational cost of the conventional implicit multisymplectic algorithms, we melt LOD idea into the multisymplectic algorithms for multi-dimensional HPDEs in the paper.

The outline of the paper is organized as follows: In Section 2, the conservation laws and the multisymplecticity are investigated for the 3-D Maxwell's equations. In Section 3, we split the 3-D Maxwell's equation into three LOD multisymplectic Hamiltonian systems. In Section 4, the LOD multisymplectic discretization is successfully applied to the split Maxwell's equations, and an LOD multisymplectic scheme (LOD-MS) is established for the sub-Hamiltonian Maxwell's equations. In Section 5, we explore the stability, convergence, dissipation, dispersion relation as well for the just established scheme. In Section 6, one-, two- and three-dimensional Maxwell's equations are simulated by the novel LOD-MS. We conclude and remark the paper in Section 7.

2. Maxwell's equations and its conservation laws

Maxwell's equations are the most foundational equations in electromagnetism and are widely applied to many application fields. They are mathematical expressions of the natural laws correlative fields, such as Ampère's law and Faraday's law, etc. Recently, in large scale and long-term computation, it is extremely crucial to propose efficient numerical methods to simulate Maxwell's equation with two or three spatial dimensions. Up to now, it is rather difficult numerically to simulate them by conventional numerical methods in a PC owning to the restriction of memory and CPU. To clean the difficulties, the ADI and the LOD numerical techniques are frequently adopted, which are often combined with finite difference time-domain (FDTD) methods [22,31]. For example, Holland considered the ADI method combined with Yee's scheme for two-dimensional transverse electric waves in [23], however, it is difficult to extend to 3-D problems. Recently, some splitting FDTD methods for two-dimensional Maxwell's equations are proposed in [24–26]. As for the 3-D problems, in [28], Sha et al. proposed symplectic-FDTD techniques, however, they are conditionally stable. Zheng et al. first proposed an unconditionally stable ADI-FDTD scheme in [27], which analyzed the stability by Fourier method. In addition, Lee and Fornberg brought up some unconditionally stable time stepping methods [29,30], which include some techniques to enhance the temporal accuracy of the schemes, such as extrapolation and deferred correction techniques, but none of them has considered the multisymplecticity of Maxwell's equations. In [16,17], Cai et al. and Su et al. considered multisymplectic schemes. However, they are all completely implicit, and are difficult in code for 3-D problems, and in their experimental work, they only performed 1-D or 2-D problems. We consider the LOD-MS for the Maxwell's equations in the paper, and overcome the pitfalls of the above numerical techniques. It is very efficient in code, and unconditionally stable, non-dissipative, and of first order convergence in time and second order in space. Furthermore, it is energy-preserving and multisymplectic.

For a linear homogeneous medium within linear isotropic material with the permittivity ϵ and the permeability μ, the scattering of electromagnetic waves without the charges or the currents can be described by the 3-D Maxwell's equations in curl formulation

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon}\nabla \times \mathbf{H}, \tag{2.1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu}\nabla \times \mathbf{E}, \tag{2.2}$$

where $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{H} = (H_x, H_y, H_z)$ represent the electric field and the magnetic field, respectively. The domain $\Omega \times [0,T] = [0,a] \times [0,b] \times [0,c] \times [0,T]$ under consideration is occupied by this medium and surrounded by perfect conductors. Therefore, the perfectly electric conducting boundary condition is imposed on the boundary, namely,

$$(\mathbf{E}, 0) \times (\vec{\mathbf{n}}, 0) = 0, \quad \text{on } \partial\Omega \times (0, T], \tag{2.3}$$

where $\vec{\mathbf{n}}$ is the outward normal vector of the boundary.

The curl equations (2.1) and (2.2) can be written into the componentwise formula

$$\frac{\partial}{\partial t}\begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{1}{\epsilon}\left(\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y\right) \\ \frac{1}{\epsilon}\left(\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z\right) \\ \frac{1}{\epsilon}\left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right) \\ \frac{1}{\mu}\left(\frac{\partial}{\partial z}E_y - \frac{\partial}{\partial y}E_z\right) \\ \frac{1}{\mu}\left(\frac{\partial}{\partial x}E_z - \frac{\partial}{\partial z}E_x\right) \\ \frac{1}{\mu}\left(\frac{\partial}{\partial y}E_x - \frac{\partial}{\partial x}E_y\right) \end{bmatrix}. \tag{2.4}$$

Let $\mathbf{z} = (H_x, H_y, H_z, E_x, E_y, E_z)^T$, the componentwise formula (2.4) is naturally multisymplectic, and the corresponding matrices are

$$M = \begin{bmatrix} \mathbf{0}_3 & -I_3 \\ I_3 & \mathbf{0}_3 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & -\frac{1}{\epsilon} & & \mathbf{0}_3 & \\ 0 & \frac{1}{\epsilon} & 0 & & & \\ & & & 0 & 0 & 0 \\ & \mathbf{0}_3 & & 0 & 0 & -\frac{1}{\mu} \\ & & & 0 & \frac{1}{\mu} & 0 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0 & 0 & \frac{1}{\epsilon} & & & \\ 0 & 0 & 0 & & \mathbf{0}_3 & \\ -\frac{1}{\epsilon} & 0 & 0 & & & \\ & & & 0 & 0 & \frac{1}{\mu} \\ & \mathbf{0}_3 & & 0 & 0 & 0 \\ & & & -\frac{1}{\mu} & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & -\frac{1}{\epsilon} & 0 & & & \\ \frac{1}{\epsilon} & 0 & 0 & & \mathbf{0}_3 & \\ 0 & 0 & 0 & & & \\ & & & 0 & -\frac{1}{\mu} & 0 \\ & \mathbf{0}_3 & & \frac{1}{\mu} & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix},$$

where $\mathbf{0}_3$ is the 3×3 zero matrix and I_3 is the 3×3 identity matrix. The Hamiltonian function is $S(\mathbf{z}) = 0$.

The MSCL of this multisymplectic HPDE (2.4) is

$$\begin{aligned} &\frac{\partial}{\partial t}(dE_x \wedge dH_x + dE_y \wedge dH_y + dE_z \wedge dH_z) + \frac{\partial}{\partial x}\left(\frac{1}{\epsilon}dH_z \wedge dH_y + \frac{1}{\mu}dE_z \wedge dE_y\right) \\ &\quad + \frac{\partial}{\partial y}\left(\frac{1}{\epsilon}dH_x \wedge dH_z + \frac{1}{\mu}dE_x \wedge dE_z\right) + \frac{\partial}{\partial z}\left(\frac{1}{\epsilon}dH_y \wedge dH_x + \frac{1}{\mu}dE_y \wedge dE_x\right) = 0. \end{aligned} \tag{2.5}$$

By (1.3), through a direct calculation, the corresponding local energy conservation law to the Maxwell's equations (2.1) and (2.2) written into the curl-divergence form is

$$\frac{\partial}{\partial t}\left(\frac{1}{\epsilon}\mathbf{H} \cdot \nabla \times \mathbf{H} + \frac{1}{\mu}\mathbf{E} \cdot \nabla \times \mathbf{E}\right) + \left(\frac{1}{\epsilon}\nabla \cdot (\mathbf{H} \times \mathbf{H}_t) + \frac{1}{\mu}\nabla \cdot (\mathbf{E} \times \mathbf{E}_t)\right) = 0. \tag{2.6}$$

Under the perfectly electric conducting boundary condition (2.3), it implies the total energy conservation law

$$\int_\Omega \left(\frac{1}{\epsilon}\mathbf{H} \cdot \nabla \times \mathbf{H} + \frac{1}{\mu}\mathbf{E} \cdot \nabla \times \mathbf{E}\right) d\Omega = \int_\Omega \left(\mathbf{H} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{H}}{\partial t}\right) d\Omega = Constant. \tag{2.7}$$

It is noted that for the first equality we have used the curl equations (2.1) and (2.2).

Furthermore, the Maxwell's equations (2.1) and (2.2) conserve the following invariants:

$$\text{Energy I}: \quad \int_{\Omega} (\epsilon |\mathbf{E}(x,t)|^2 + \mu |\mathbf{H}(x,t)|^2)\, d\Omega = Constant, \tag{2.8}$$

$$\text{Energy II}: \quad \int_{\Omega} \left(\epsilon \left| \frac{\partial \mathbf{E}(x,t)}{\partial t} \right|^2 + \mu \left| \frac{\partial \mathbf{H}(x,t)}{\partial t} \right|^2 \right) d\Omega = Constant. \tag{2.9}$$

The first invariant (2.8) is called Poynting theorem in electromagnetism and can be easily verified, and the second one is a little more complex. For more details, see [24].

In the two-dimensional transverse magnetic (TM) polarization case, the electric field and magnetic field read $\mathbf{E} = (0,0,E_z)^T$, $\mathbf{H} = (H_x, H_y, 0)^T$. Therefore, the Maxwell's equations (2.1) and (2.2) become

$$\begin{cases} \frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right), \\ \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \frac{\partial E_z}{\partial x}, \\ \frac{\partial H_x}{\partial t} = -\frac{1}{\mu} \frac{\partial E_z}{\partial y}. \end{cases} \tag{2.10}$$

3. Discretization of sub-Hamiltonian PDEs

For a Hamiltonian ODE, one of the most important methods to construct symplectic integrators is the vector field splitting method [3]. For example, assume that the Hamiltonian function can be split into $H(z) = H_1(z) + H_2(z) + \cdots + H_m(z)$, and the Hamiltonian system

$$z_t = J \nabla_z H(z) = J \nabla (H_1(z) + H_2(z) + \cdots + H_m(z)), \tag{3.1}$$

can be split into m subsystems

$$z_t = J \nabla_z H_j(z), \quad j = 1, 2, \ldots, m, \tag{3.2}$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix.

To numerically solve the Hamiltonian system (3.1), we firstly solve subsystems (3.2) which are easier than directly solving system (3.1), one after another by symplectic integrators. The numerical solution of one subsystem is employed as the initial values of the next one (see e.g. [2,3] and references therein). However, for the multisymplectic Hamiltonian system (1.1), to our knowledge, there is *no* study on its splitting methods. We explore its splitting multisymplectic integrators in the section. We firstly localize the original multisymplectic HPDE to several one-dimensional ones, then consider the multisymplectic discretization for the LOD HPDEs.

For the general m-dimensional multisymplectic Hamiltonian system (1.1), let us consider the LOD multisymplectic Hamiltonian system

$$\frac{1}{m} M \mathbf{z}_t + K_k \mathbf{z}_{x_k} = \nabla_{\mathbf{z}} S_k(\mathbf{z}), \quad k = 1, 2, \ldots, m, \tag{3.3}$$

where $S_k(\mathbf{z})$ can be any splitting of $S(\mathbf{z})$, but it ought to satisfy $\sum_{k=1}^{m} S_k(\mathbf{z}) = S(\mathbf{z})$.

Certainly, the LOD multisymplectic Hamiltonian systems satisfy the LOD MSCLs

$$\frac{1}{m} \frac{\partial}{\partial t} \omega + \frac{\partial}{\partial x_j} \kappa_j = 0, \quad j = 1, 2, \ldots, m. \tag{3.4}$$

It can be verified that the sum of the above m conservation laws is just the MSCL (1.2).

For example, the multisymplectic Hamiltonian system (2.4) can be split into the following three LOD subsystems

$$\frac{1}{3} M \frac{\partial}{\partial t} \mathbf{z} + K_k \frac{\partial}{\partial x_k} \mathbf{z} = 0, \quad k = 1, 2, 3, \tag{3.5}$$

where M, K_1, K_2, K_3 are the same as those presented previously. The corresponding LOD MSCLs for (3.5) read

$$\frac{1}{3} \frac{\partial}{\partial t} (dE_x \wedge dH_x + dE_y \wedge dH_y + dE_z \wedge dH_z) + \frac{\partial}{\partial x} \left(\frac{1}{\epsilon} dH_z \wedge dH_y + \frac{1}{\mu} dE_z \wedge dE_y \right) = 0, \tag{3.6}$$

$$\frac{1}{3} \frac{\partial}{\partial t} (dE_x \wedge dH_x + dE_y \wedge dH_y + dE_z \wedge dH_z) + \frac{\partial}{\partial y} \left(\frac{1}{\epsilon} dH_x \wedge dH_z + \frac{1}{\mu} dE_x \wedge dE_z \right) = 0, \tag{3.7}$$

$$\frac{1}{3} \frac{\partial}{\partial t} (dE_x \wedge dH_x + dE_y \wedge dH_y + dE_z \wedge dH_z) + \frac{\partial}{\partial z} \left(\frac{1}{\epsilon} dH_y \wedge dH_x + \frac{1}{\mu} dE_y \wedge dE_x \right) = 0. \tag{3.8}$$

In fact, the multisymplectic subsystems (3.5) can be cut down

$$\frac{1}{2}\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}H_y\\H_z\\E_y\\E_z\end{bmatrix}_t+\begin{bmatrix}0&-\frac{1}{\epsilon}&0&0\\\frac{1}{\epsilon}&0&0&0\\0&0&0&-\frac{1}{\mu}\\0&0&\frac{1}{\mu}&0\end{bmatrix}\begin{bmatrix}H_y\\H_z\\E_y\\E_z\end{bmatrix}_x=0, \tag{3.9}$$

$$\frac{1}{2}\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}H_x\\H_z\\E_x\\E_z\end{bmatrix}_t+\begin{bmatrix}0&\frac{1}{\epsilon}&0&0\\-\frac{1}{\epsilon}&0&0&0\\0&0&0&\frac{1}{\mu}\\0&0&-\frac{1}{\mu}&0\end{bmatrix}\begin{bmatrix}H_x\\H_z\\E_x\\E_z\end{bmatrix}_y=0, \tag{3.10}$$

$$\frac{1}{2}\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}H_x\\H_y\\E_x\\E_y\end{bmatrix}_t+\begin{bmatrix}0&-\frac{1}{\epsilon}&0&0\\\frac{1}{\epsilon}&0&0&0\\0&0&0&-\frac{1}{\mu}\\0&0&\frac{1}{\mu}&0\end{bmatrix}\begin{bmatrix}H_x\\H_y\\E_x\\E_y\end{bmatrix}_z=0. \tag{3.11}$$

Thus, the MSCLs (3.6)–(3.8) are reduced to

$$\frac{1}{2}\frac{\partial}{\partial t}(dE_y\wedge dH_y+dE_z\wedge dH_z)+\frac{\partial}{\partial x}\left(\frac{1}{\epsilon}dH_z\wedge dH_y+\frac{1}{\mu}dE_z\wedge dE_y\right)=0, \tag{3.12}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(dE_x\wedge dH_x+dE_z\wedge dH_z)+\frac{\partial}{\partial y}\left(\frac{1}{\epsilon}dH_x\wedge dH_z+\frac{1}{\mu}dE_x\wedge dE_z\right)=0, \tag{3.13}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(dE_x\wedge dH_x+dE_y\wedge dH_y)+\frac{\partial}{\partial z}\left(\frac{1}{\epsilon}dH_y\wedge dH_x+\frac{1}{\mu}dE_y\wedge dE_x\right)=0. \tag{3.14}$$

The multisymplectic subsystems (3.9)–(3.11) can be uniformly written in the form

$$\overline{M}\bar{\mathbf{z}}_t+\overline{K}\bar{\mathbf{z}}_{x_k}=0,\quad k=1,2,3, \tag{3.15}$$

where

$$\overline{M}=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad \overline{K}=\begin{bmatrix}0&\pm\frac{1}{\epsilon}&0&0\\\mp\frac{1}{\epsilon}&0&0&0\\0&0&0&\pm\frac{1}{\mu}\\0&0&\mp\frac{1}{\mu}&0\end{bmatrix},$$

$$\bar{\mathbf{z}}=\begin{bmatrix}H_y\\H_z\\E_y\\E_z\end{bmatrix}\quad\text{or}\quad\bar{\mathbf{z}}=\begin{bmatrix}H_x\\H_z\\E_x\\E_z\end{bmatrix}\quad\text{or}\quad\bar{\mathbf{z}}=\begin{bmatrix}H_x\\H_y\\E_x\\E_y\end{bmatrix}.$$

For the two-dimensional Maxwell's equation (2.10), it can be split into

$$\begin{cases}\frac{1}{2}\frac{\partial E_z}{\partial t}=\frac{1}{\epsilon}\frac{\partial H_y}{\partial x},\\ \frac{\partial H_y}{\partial t}=\frac{1}{\mu}\frac{\partial E_z}{\partial x},\end{cases}\quad\text{and}\quad\begin{cases}\frac{1}{2}\frac{\partial E_z}{\partial t}=-\frac{1}{\epsilon}\frac{\partial H_x}{\partial y},\\ \frac{\partial H_x}{\partial t}=-\frac{1}{\mu}\frac{\partial E_z}{\partial y}.\end{cases} \tag{3.16}$$

In what follows, we investigate the directional multisymplectic numerical methods for the multisymplectic HPDEs (1.1) and (3.3). In other words, numerical integrator preserves LOD multisymplectic conservation laws which is consisted of symplectic structures under considering spatial direction and temporal direction. For simplicity, we first discuss the multisymplectic numerical method for (3.3).

Let h_x, h_y and h_z be the mesh sizes along x, y and z directions, respectively, and τ the time step length. The spatial–temporal domain $[x_L,x_R]\times[y_L,y_R]\times[z_L,z_R]\times[0,T]$ is partitioned by parallel lines, $x_i=x_L+ih_x,\ y_j=y_L+jh_y,\ z_k=z_L+kh_z,\ t^n=n\tau$, for $i=0,1,\ldots,I;\ j=0,1,2,\ldots,J;\ k=0,1,2,\ldots,K$ and $n=0,1,\ldots,N$. The grid point function $\psi^n_{i,j,k}$ is the approximation of $\psi(x,y,z,t)$ at nodes (x_i,y_j,z_k,t^n). The general difference operators are employed:

$$\delta_t\psi^n_{i,j,k}=\frac{\psi^{n+1}_{i,j,k}-\psi^n_{i,j,k}}{\tau},\quad \delta_x\psi^n_{i,j,k}=\frac{\psi^n_{i+1,j,k}-\psi^n_{i,j,k}}{h_x},$$

$$\delta_y\psi^n_{i,j,k}=\frac{\psi^n_{i,j+1,k}-\psi^n_{i,j,k}}{h_y},\quad \delta_z\psi^n_{i,j,k}=\frac{\psi^n_{i,j,k+1}-\psi^n_{i,j,k}}{h_z}.$$

Under the cuboid spatial domain and the uniform mesh division, the difference operators are commutable, namely, $\delta_\alpha\delta_\beta\psi^n_{i,j,k}=\delta_\beta\delta_\alpha\psi^n_{i,j,k}$, where α and β can be taken either of the directions x, y, z and t.

Applying an s-stage and an r-stage Runge–Kutta methods to the LOD Hamiltonian system (3.3) in the t-direction and x_α direction, respectively, we have

$$\begin{cases} Z_i^m = z_i^0 + \tau \sum_{n=1}^{s} a_{mn}\partial_t Z_i^n, \quad m = 1,2,\dots,s, \\ z_i^1 = z_i^0 + \tau \sum_{m=1}^{s} b_m \partial_t Z_i^m, \\ Z_i^m = z_0^m + h_\alpha \sum_{j=1}^{r} \bar{a}_{ij}\partial_\alpha Z_j^m, \quad i = 1,2,\dots,r, \\ z_1^m = z_0^m + h_\alpha \sum_{i=1}^{r} \bar{b}_i \partial_\alpha Z_i^m, \quad \alpha = x,y,z, \\ M\partial_t Z_i^m + K\partial_\alpha Z_i^m = \nabla_z S(Z_i^m), \end{cases} \tag{3.17}$$

here we made use of the notations: $Z_p^m \approx z(\bar{c}_p h_\alpha, c_m\tau)$, $z_p^0 \approx z(\bar{c}_p h_\alpha, 0)$, $z_p^1 \approx z(\bar{c}_p h_\alpha, \tau)$, $z_0^m \approx z(0, c_m\tau)$, $z_1^m \approx z(h_\alpha, c_m\tau)$, $\partial_t Z_p^m \approx \partial_t z(\bar{c}_p h_\alpha, c_m\tau)$, $\partial_\alpha Z_p^m \approx \partial_\alpha z(\bar{c}_p h_\alpha, c_m\tau)$, and let $c_m = \sum_{n=1}^{s} a_{mn}$, $\bar{c}_p = \sum_{q=1}^{r} \bar{a}_{pq}$.

Lemma 1 (Reich [4]). *Let the multisymplectic formulation (3.3) be discretized in the t-direction and x_α-direction as (3.17). Moreover, the Runge–Kutta coefficients $\{a_{mn}\}\{b_m\}$ and $\{\bar{a}_{pq}\}\{\bar{b}_p\}$ satisfy the symplectic conditions*

$$b_m b_n - b_m a_{mn} - b_n a_{nm} = 0, \quad m,\ n = 1,2,\dots,s, \tag{3.18}$$

$$\bar{b}_p \bar{b}_q - \bar{b}_p \bar{a}_{pq} - \bar{b}_q \bar{a}_{qp} = 0, \quad p,\ q = 1,2,\dots,r, \tag{3.19}$$

there results in a multisymplectic integrator.

For $r = s = 1$, the multisymplectic Runge–Kutta method (3.17) can be written as

$$\frac{1}{m} M \frac{z_{i+\frac12}^{n+1} - z_{i+\frac12}^{n}}{\tau} + K_k \frac{z_{i+1}^{n+\frac12} - z_{i}^{n+\frac12}}{h_\alpha} = \nabla_z S_k\left(z_{i+\frac12}^{n+\frac12}\right), \quad k = 1,2,\dots,m, \tag{3.20}$$

where $z_{i+\frac12}^{n+\frac12} = \frac12\left(z_{i+1}^{n+\frac12} + z_i^{n+\frac12}\right) = \frac12\left(z_{i+\frac12}^{n+1} + z_{i+\frac12}^{n}\right) = \frac14\left(z_{i+1}^{n+1} + z_i^{n+1} + z_{i+1}^{n} + z_i^{n}\right)$. This scheme is just the frequently used Preissman scheme or central box scheme (also see [8,9,12]).

4. LOD multisymplectic method for Maxwell's equation

The Maxwell's equations (2.1) and (2.2) are 3-D HPDEs. In numerical computation, an explicit numerical method (e.g. Yee-method) suffers from a strict CFL condition, and we are often required implicit methods to simulate them which are usually unconditionally stable. However, for conventional implicit methods, they are often impracticable because of the limitation of memory and CPU. It is necessary to adopt the LOD or the ADI techniques to numerically simulate them [19–21,24,27,29,32]. To the specific structure of Maxwell's equations, it is a very suitable application LOD technique to them.

To illustrate the terrible computational cost of the non-LOD central box scheme for 3-D Maxwell's equations (2.1) and (2.2), we firstly apply central box scheme to them and have

$$\begin{cases} \delta_t E^n_{x_{i+\frac12,j+\frac12,k+\frac12}} - \frac{1}{\epsilon}\left(\delta_y H^{n+\frac12}_{z_{i+\frac12,j,k+\frac12}} - \delta_z H^{n+\frac12}_{y_{i+\frac12,j+\frac12,k}}\right) = 0, \\ \delta_t E^n_{y_{i+\frac12,j+\frac12,k+\frac12}} - \frac{1}{\epsilon}\left(\delta_z H^{n+\frac12}_{x_{i+\frac12,j+\frac12,k}} - \delta_x H^{n+\frac12}_{z_{i,j+\frac12,k+\frac12}}\right) = 0, \\ \delta_t E^n_{z_{i+\frac12,j+\frac12,k+\frac12}} - \frac{1}{\epsilon}\left(\delta_x H^{n+\frac12}_{y_{i,j+\frac12,k+\frac12}} - \delta_y H^{n+\frac12}_{x_{i+\frac12,j,k+\frac12}}\right) = 0, \\ \delta_t H^n_{x_{i+\frac12,j+\frac12,k+\frac12}} - \frac{1}{\mu}\left(\delta_z E^{n+\frac12}_{y_{i+\frac12,j+\frac12,k}} - \delta_y E^{n+\frac12}_{z_{i+\frac12,j+\frac12,k}}\right) = 0, \\ \delta_t H^n_{y_{i+\frac12,j+\frac12,k+\frac12}} - \frac{1}{\mu}\left(\delta_x E^{n+\frac12}_{z_{i,j+\frac12,k+\frac12}} - \delta_z E^{n+\frac12}_{x_{i+\frac12,j+\frac12,k}}\right) = 0, \\ \delta_t H^n_{z_{i+\frac12,j+\frac12,k+\frac12}} - \frac{1}{\mu}\left(\delta_y E^{n+\frac12}_{x_{i+\frac12,j,k+\frac12}} - \delta_x E^{n+\frac12}_{y_{i,j+\frac12,k+\frac12}}\right) = 0, \end{cases} \tag{4.1}$$

for $i = 1,2,\dots,I;\ j = 1,2,\dots,J;\ k = 1,2,\dots,K;\ n = 0,1,2,\dots$ This scheme is exactly multisymplectic and convergent to the curl equations (2.1) and (2.2) with second order in both space and time directions. Nevertheless, to solve the difference equations, we need to resolve a linear algebraic equations with $6 \times I \times J \times K$ scale. The bandwidth of its coefficient matrix is very wide. The computational cost is so intensive that we can say it is unresolvable in the computational sense of PC. As a compromise, we resort to the LOD multisymplectic integrator.

Now, we apply the central box scheme (3.20) to discretize the sub-Hamiltonian systems (3.9)–(3.11), respectively, and get the following directional multisymplectic scheme:

$$\begin{cases} \frac{1}{2}\frac{1}{\tau/2}\left(E^*_{y_{i+\frac{1}{2},j,k}} - E^n_{y_{i+\frac{1}{2},j,k}}\right) + \frac{1}{\epsilon h_x}\left(H^{n+\frac{1}{2}*}_{z_{i+1,j,k}} - H^{n+\frac{1}{2}*}_{z_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(E^*_{z_{i+\frac{1}{2},j,k}} - E^n_{z_{i+\frac{1}{2},j,k}}\right) - \frac{1}{\epsilon h_x}\left(H^{n+\frac{1}{2}*}_{y_{i+1,j,k}} - H^{n+\frac{1}{2}*}_{y_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(H^*_{y_{i+\frac{1}{2},j,k}} - H^n_{y_{i+\frac{1}{2},j,k}}\right) - \frac{1}{\mu h_x}\left(E^{n+\frac{1}{2}*}_{z_{i+1,j,k}} - E^{n+\frac{1}{2}*}_{z_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(H^*_{z_{i+\frac{1}{2},j,k}} - H^n_{z_{i+\frac{1}{2},j,k}}\right) + \frac{1}{\mu h_x}\left(E^{n+\frac{1}{2}*}_{y_{i+1,j,k}} - E^{n+\frac{1}{2}*}_{y_{i,j,k}}\right) = 0, \end{cases} \tag{4.2}$$

$$\begin{cases} \frac{1}{2}\frac{1}{\tau/2}\left(E^*_{x_{i,j+\frac{1}{2},k}} - E^n_{x_{i,j+\frac{1}{2},k}}\right) - \frac{1}{\epsilon h_y}\left(H^{n+\frac{1}{2}}_{z_{i,j+1,k}} - H^{n+\frac{1}{2}}_{z_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(E^{n+1}_{z_{i,j+\frac{1}{2},k}} - E^*_{z_{i,j+\frac{1}{2},k}}\right) + \frac{1}{\epsilon h_y}\left(H^{n+\frac{1}{2}*}_{x_{i,j+1,k}} - H^{n+\frac{1}{2}*}_{x_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(H^*_{x_{i,j+\frac{1}{2},k}} - H^n_{x_{i,j+\frac{1}{2},k}}\right) + \frac{1}{\mu h_y}\left(E^{n+\frac{1}{2}}_{z_{i,j+1,k}} - E^{n+\frac{1}{2}}_{z_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(H^{n+1}_{z_{i,j+\frac{1}{2},k}} - H^*_{z_{i,j+\frac{1}{2},k}}\right) - \frac{1}{\mu h_y}\left(E^{n+\frac{1}{2}*}_{x_{i,j+1,k}} - E^{n+\frac{1}{2}*}_{x_{i,j,k}}\right) = 0, \end{cases} \tag{4.3}$$

$$\begin{cases} \frac{1}{2}\frac{1}{\tau/2}\left(E^{n+1}_{x_{i,j,k+\frac{1}{2}}} - E^*_{x_{i,j,k+\frac{1}{2}}}\right) + \frac{1}{\epsilon h_z}\left(H^{n+\frac{1}{2}}_{y_{i,j,k+1}} - H^{n+\frac{1}{2}}_{y_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(E^{n+1}_{y_{i,j,k+\frac{1}{2}}} - E^*_{y_{i,j,k+\frac{1}{2}}}\right) - \frac{1}{\epsilon h_z}\left(H^{n+\frac{1}{2}}_{x_{i,j,k+1}} - H^{n+\frac{1}{2}}_{x_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(H^{n+1}_{x_{i,j,k+\frac{1}{2}}} - H^*_{x_{i,j,k+\frac{1}{2}}}\right) - \frac{1}{\mu h_z}\left(E^{n+\frac{1}{2}}_{y_{i,j,k+1}} - E^{n+\frac{1}{2}}_{y_{i,j,k}}\right) = 0, \\ \frac{1}{2}\frac{1}{\tau/2}\left(H^{n+1}_{y_{i,j,k+\frac{1}{2}}} - H^*_{y_{i,j,k+\frac{1}{2}}}\right) + \frac{1}{\mu h_z}\left(E^{n+\frac{1}{2}}_{x_{i,j,k+1}} - E^{n+\frac{1}{2}}_{x_{i,j,k}}\right) = 0, \end{cases} \tag{4.4}$$

where U^* is an intermediate value between U^n and U^{n+1}, $U^{n+\frac{1}{2}*}_{i,j,k} = \frac{1}{2}\left(U^n_{i,j,k} + U^*_{i,j,k}\right)$, $U^{n+\frac{1}{2}}_{i,j,k} = \frac{1}{2}\left(U^{n+1}_{i,j,k} + U^*_{i,j,k}\right)$.

It is obviously that the numerical integrators (4.2)–(4.4) satisfy the discrete analogue of LOD MSCL (3.12)–(3.14), that is,

$$\frac{\omega^*_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} - \omega^n_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}}{\tau} + \frac{\kappa^{n+\frac{1}{2}*}_{x_{i+1,j+\frac{1}{2},k+\frac{1}{2}}} - \kappa^{n+\frac{1}{2}*}_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}}{h_x} = 0, \tag{4.5}$$

$$\frac{\omega^{n+1*}_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} - \omega^{n*}_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}}{\tau} + \frac{\kappa^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j+1,k+\frac{1}{2}}} - \kappa^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}}{h_y} = 0, \tag{4.6}$$

$$\frac{\omega^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} - \omega^*_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}}{\tau} + \frac{\kappa^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2},k+1}} - \kappa^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}}{h_z} = 0, \tag{4.7}$$

where

$$\begin{aligned}
\omega^*_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\left(dE^*_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^*_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} + dE^*_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^*_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}\right), \\
\omega^n_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\left(dE^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} + dE^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}\right), \\
\kappa^{n+\frac{1}{2}*}_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{\epsilon} dH^{n+\frac{1}{2}*}_{z_{i,j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^{n+\frac{1}{2}*}_{y_{i,j+\frac{1}{2},k+\frac{1}{2}}} + \frac{1}{\mu} dE^{n+\frac{1}{2}*}_{z_{i,j+\frac{1}{2},k+\frac{1}{2}}} \wedge dE^{n+\frac{1}{2}*}_{y_{i,j+\frac{1}{2},k+\frac{1}{2}}}, \\
\omega^{n+1*}_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\left(dE^*_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^*_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} + dE^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}\right), \\
\omega^{n*}_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\left(dE^n_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^n_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} + dE^*_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^*_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}\right), \\
\kappa^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} &= \frac{1}{\epsilon} dH^{n+\frac{1}{2}*}_{x_{i+\frac{1}{2},j,k+\frac{1}{2}}} \wedge dH^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j,k+\frac{1}{2}}} + \frac{1}{\mu} dE^{n+\frac{1}{2}*}_{x_{i+\frac{1}{2},j,k+\frac{1}{2}}} \wedge dE^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j,k+\frac{1}{2}}}, \\
\omega^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\left(dE^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} + dE^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}\right), \\
\omega^*_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\left(dE^*_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^*_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} + dE^*_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \wedge dH^*_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}\right), \\
\kappa^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}} &= \frac{1}{\epsilon} dH^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j+\frac{1}{2},k}} \wedge dH^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j+\frac{1}{2},k}} + \frac{1}{\mu} dE^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j+\frac{1}{2},k}} \wedge dE^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j+\frac{1}{2},k}}.
\end{aligned}$$

Consequently, schemes (4.2)–(4.4) are LOD-MS.

Adding the above directional discrete multisymplectic conservation law altogether, it yields

$$\frac{\omega^{n+1}_{i+\frac12,j+\frac12,k+\frac12}-\omega^{n}_{i+\frac12,j+\frac12,k+\frac12}}{\tau}+\frac{\kappa^{n+\frac12*}_{x_{i+1,j+\frac12,k+\frac12}}-\kappa^{n+\frac12*}_{x_{i,j+\frac12,k+\frac12}}}{h_x}+\frac{\kappa^{n+\frac12*}_{y_{i+\frac12,j+1,k+\frac12}}-\kappa^{n+\frac12*}_{y_{i+\frac12,j,k+\frac12}}}{h_y}+\frac{\kappa^{n+\frac12}_{z_{i+\frac12,j+\frac12,k+1}}-\kappa^{n+\frac12}_{z_{i+\frac12,j+\frac12,k}}}{h_z}=0, \tag{4.8}$$

where $\omega^n_{i+\frac12,j+\frac12,k+\frac12}=dE^n_{x_{i+\frac12,j+\frac12,k+\frac12}}\wedge dH^n_{x_{i+\frac12,j+\frac12,k+\frac12}}+dE^n_{y_{i+\frac12,j+\frac12,k+\frac12}}\wedge dH^n_{y_{i+\frac12,j+\frac12,k+\frac12}}+dE^n_{z_{i+\frac12,j+\frac12,k+\frac12}}\wedge dH^n_{z_{i+\frac12,j+\frac12,k+\frac12}}$. This is a discrete analogue of the MSCL (2.5).

The local MSCL (4.8) is naturally global if the solutions are periodic, in other words, the global symplectic conservation law

$$\sum_{i,j,k}\omega^{n+1}_{i+\frac12,j+\frac12,k+\frac12}=\sum_{i,j,k}\omega^{n}_{i+\frac12,j+\frac12,k+\frac12}, \tag{4.9}$$

is held. This means that LOD-MS is at least symplectic structure-preserving, although it may be not exactly multisymplectic preserving in some situation.

The schemes (4.2)–(4.4) can be rearranged into,

$$\begin{cases}\frac{1}{\tau}\left(E^*_{y_{i+\frac12,j,k}}-E^n_{y_{i+\frac12,j,k}}\right)+\frac{1}{2\epsilon h_x}\left[\left(H^*_{z_{i+1,j,k}}-H^*_{z_{i,j,k}}\right)+\left(H^n_{z_{i+1,j,k}}-H^n_{z_{i,j,k}}\right)\right]=0,\\ \frac{1}{\tau}\left(H^*_{z_{i+\frac12,j,k}}-H^n_{z_{i+\frac12,j,k}}\right)+\frac{1}{2\mu h_x}\left[\left(E^*_{y_{i+1,j,k}}-E^*_{y_{i,j,k}}\right)+\left(E^n_{y_{i+1,j,k}}-E^n_{y_{i,j,k}}\right)\right]=0,\end{cases} \tag{4.10}$$

$$\begin{cases}\frac{1}{\tau}\left(E^*_{z_{i+\frac12,j,k}}-E^n_{z_{i+\frac12,j,k}}\right)-\frac{1}{2\epsilon h_x}\left[\left(H^*_{y_{i+1,j,k}}-H^*_{y_{i,j,k}}\right)+\left(H^n_{y_{i+1,j,k}}-H^n_{y_{i,j,k}}\right)\right]=0,\\ \frac{1}{\tau}\left(H^*_{y_{i+\frac12,j,k}}-H^n_{y_{i+\frac12,j,k}}\right)-\frac{1}{2\mu h_x}\left[\left(E^*_{z_{i+1,j,k}}-E^*_{z_{i,j,k}}\right)+\left(E^n_{z_{i+1,j,k}}-E^n_{z_{i,j,k}}\right)\right]=0,\end{cases} \tag{4.11}$$

$$\begin{cases}\frac{1}{\tau}\left(E^*_{x_{i,j+\frac12,k}}-E^n_{x_{i,j+\frac12,k}}\right)-\frac{1}{2\epsilon h_y}\left[\left(H^{n+1}_{z_{i,j+1,k}}-H^{n+1}_{z_{i,j,k}}\right)+\left(H^*_{z_{i,j+1,k}}-H^*_{z_{i,j,k}}\right)\right]=0,\\ \frac{1}{\tau}\left(H^{n+1}_{z_{i,j+\frac12,k}}-H^*_{z_{i,j+\frac12,k}}\right)-\frac{1}{2\mu h_y}\left[\left(E^*_{x_{i,j+1,k}}-E^*_{x_{i,j,k}}\right)+\left(E^n_{x_{i,j+1,k}}-E^n_{x_{i,j,k}}\right)\right]=0,\end{cases} \tag{4.12}$$

$$\begin{cases}\frac{1}{\tau}\left(E^{n+1}_{z_{i,j+\frac12,k}}-E^*_{z_{i,j+\frac12,k}}\right)+\frac{1}{2\epsilon h_y}\left[\left(H^*_{x_{i,j+1,k}}-H^*_{x_{i,j,k}}\right)+\left(H^n_{x_{i,j+1,k}}-H^n_{x_{i,j,k}}\right)\right]=0,\\ \frac{1}{\tau}\left(H^*_{x_{i,j+\frac12,k}}-H^n_{x_{i,j+\frac12,k}}\right)+\frac{1}{2\mu h_y}\left[\left(E^{n+1}_{z_{i,j+1,k}}-E^{n+1}_{z_{i,j,k}}\right)+\left(E^*_{z_{i,j+1,k}}-E^*_{z_{i,j,k}}\right)\right]=0,\end{cases} \tag{4.13}$$

$$\begin{cases}\frac{1}{\tau}\left(E^{n+1}_{x_{i,j,k+\frac12}}-E^*_{x_{i,j,k+\frac12}}\right)+\frac{1}{2\epsilon h_z}\left[\left(H^{n+1}_{y_{i,j,k+1}}-H^{n+1}_{y_{i,j,k}}\right)+\left(H^*_{y_{i,j,k+1}}-H^*_{y_{i,j,k}}\right)\right]=0,\\ \frac{1}{\tau}\left(H^{n+1}_{y_{i,j,k+\frac12}}-H^*_{y_{i,j,k+\frac12}}\right)+\frac{1}{2\mu h_z}\left[\left(E^{n+1}_{x_{i,j,k+1}}-E^{n+1}_{x_{i,j,k}}\right)+\left(E^*_{x_{i,j,k+1}}-E^*_{x_{i,j,k}}\right)\right]=0,\end{cases} \tag{4.14}$$

$$\begin{cases}\frac{1}{\tau}\left(E^{n+1}_{y_{i,j,k+\frac12}}-E^*_{y_{i,j,k+\frac12}}\right)-\frac{1}{2\epsilon h_z}\left[\left(H^{n+1}_{x_{i,j,k+1}}-H^{n+1}_{x_{i,j,k}}\right)+\left(H^*_{x_{i,j,k+1}}-H^*_{x_{i,j,k}}\right)\right]=0,\\ \frac{1}{\tau}\left(H^{n+1}_{x_{i,j,k+\frac12}}-H^*_{x_{i,j,k+\frac12}}\right)-\frac{1}{2\mu h_z}\left[\left(E^{n+1}_{y_{i,j,k+1}}-E^{n+1}_{y_{i,j,k}}\right)+\left(E^*_{y_{i,j,k+1}}-E^*_{y_{i,j,k}}\right)\right]=0.\end{cases} \tag{4.15}$$

From (4.10)–(4.15), we can see that the LOD-MS can be coded easily and memory economic. In every marching time step, we only need to solve some one-dimensional scale linear algebraic systems. Moreover, there is no extra memory wanted to save the intermediate variables. In the practical performance of the scheme, it need not solve components coupled equations. Indeed, from the above three steps, we can use the second equation to eliminate the magnetic component on the new time level at the first equation, and get a tridiagonal system with respect to electric component, then substitute the new electric component into the second equation, and a two-diagonal system with respect to magnetic component is obtained. For example, for Eq. (4.10), we substitute the expression of $H^*_{z_{i+\frac12,j,k}}$ into the first equation, then turn against solving $H^*_{z_{i,j,k}}$, that is in the following form:

$$\begin{cases}\left(E^*_{y_{i-1,j,k}}+2E^*_{y_{i,j,k}}+E^*_{y_{i+1,j,k}}\right)-\epsilon_x\mu_x\left(E^*_{y_{i-1,j,k}}-2E^*_{y_{i,j,k}}+E^*_{y_{i+1,j,k}}\right)\\ \quad=\left(E^n_{y_{i-1,j,k}}+2E^n_{y_{i,j,k}}+E^n_{y_{i+1,j,k}}\right)+\epsilon_x\mu_x\left(E^n_{y_{i-1,j,k}}-2E^n_{y_{i,j,k}}+E^n_{y_{i+1,j,k}}\right)-2\epsilon_x\left(H^n_{z_{i+1,j,k}}-H^n_{z_{i-1,j,k}}\right),\\ H^*_{z_{i+\frac12,j,k}}=H^n_{z_{i+\frac12,j,k}}-\frac12\mu_x\left(E^*_{y_{i+1,j,k}}-E^*_{y_{i,j,k}}\right)-\frac12\mu_x\left(E^n_{y_{i+1,j,k}}-E^n_{y_{i,j,k}}\right),\end{cases} \tag{4.16}$$

Unfortunately, for periodic problems, these formulations are only suitable for odd I, J and K since the coefficient matrices corresponding to the second difference equations are singular for even ones, that is

$$\begin{vmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{vmatrix}_n = \begin{cases} 2, & n \text{ isodd}, \\ 0, & n \text{ iseven}. \end{cases}$$

For the two-dimensional Maxwell's equation (2.10), the conventional multisymplectic central box scheme is

$$\begin{cases} \delta_t E^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{\epsilon}\left(\delta_x H^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j}} - \delta_y H^{n+\frac{1}{2}}_{x_{i,j+\frac{1}{2}}}\right), \\ \delta_t H^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}} = -\frac{1}{\mu}\delta_y E^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j}}, \\ \delta_t H^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{\mu}\delta_x E^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2}}}, \end{cases} \tag{4.17}$$

and the LOD-MS reads

$$\begin{cases} \frac{1}{\tau}\left(E^*_{z_{i+\frac{1}{2},j}} - E^n_{z_{i+\frac{1}{2},j}}\right) = \frac{1}{2\epsilon}\left(\delta_x H^n_{y_{i+\frac{1}{2},j}} + \delta_x H^n_{y_{i+\frac{1}{2},j}}\right), \\ \frac{1}{\tau}\left(H^{n+1}_{y_{i+\frac{1}{2},j}} - H^n_{y_{i+\frac{1}{2},j}}\right) = \frac{1}{2\mu}\left(\delta_x E^{n+1}_{z_{i+\frac{1}{2},j}} + \delta_x E^*_{z_{i+\frac{1}{2},j}}\right), \end{cases} \tag{4.18}$$

$$\begin{cases} \frac{1}{\tau}\left(E^{n+1}_{z_{i,j+\frac{1}{2}}} - E^*_{z_{i,j+\frac{1}{2}}}\right) = -\frac{1}{2\epsilon}\left(\delta_y H^n_{x_{i+\frac{1}{2},j}} + \delta_y H^{n+1}_{x_{i+\frac{1}{2},j}}\right), \\ \frac{1}{\tau}\left(H^{n+1}_{x_{i,j+\frac{1}{2}}} - H^n_{x_{i,j+\frac{1}{2}}}\right) = -\frac{1}{2\mu}\left(\delta_y E^*_{z_{i+\frac{1}{2},j}} + \delta_y E^{n+1}_{z_{i+\frac{1}{2},j}}\right). \end{cases} \tag{4.19}$$

Even if in this case, the conventional multisymplectic box scheme (4.17) is hard in computing, and the LOD-MS (4.18) and (4.19) can be performed easily. Moreover, it is amenable to parallel processing.

5. Stability, dissipation, dispersion and convergence analysis

In the previous section, we have observed that the new LOD-MS is economic and simple in coding. Furthermore, it is not only totally symplectic structure-preserving, but also LOD multisymplectic structure-preserving. In this section, we will find that the scheme is unconditionally stable and non-dissipative. The convergence, dispersive relation as well will be analyzed.

Let $\epsilon_x = \frac{\tau}{\epsilon h_x}$, $\epsilon_y = \frac{\tau}{\epsilon h_y}$, $\epsilon_z = \frac{\tau}{\epsilon h_z}$, $\mu_x = \frac{\tau}{\mu h_x}$, $\mu_y = \frac{\tau}{\mu h_y}$, $\mu_z = \frac{\tau}{\mu h_z}$. To the above goals, we rewrite the LOD-MS (4.10)–(4.15) in a uniformly formulate

$$\begin{bmatrix} A_1 & \pm\epsilon_s A_2 \\ \pm\mu_s A_2 & A_1 \end{bmatrix}\begin{bmatrix} E^{n+1} \\ H^{n+1} \end{bmatrix} = \begin{bmatrix} A_1 & \mp\epsilon_s A_2 \\ \mp\mu_s A_2 & A_1 \end{bmatrix}\begin{bmatrix} E^n \\ H^n \end{bmatrix}, \tag{5.1}$$

where $s = x$ or y or z, $[E^n, \quad H^n]^T$ is a vector whose components consist of a component of electric field E and a component of magnetic field H, and

$$A_1 = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & -1 \end{bmatrix}.$$

Since the LOD-MS (4.10)–(4.15) are all locally one-dimensional schemes, we first study the stability of the one-dimensional problems. We take scheme (4.10) for instance. Now let

$$\begin{bmatrix} E^n_{y_{i,j,k}} \\ H^n_{z_{i,j,k}} \end{bmatrix} = \begin{bmatrix} E_0 \\ H_0 \end{bmatrix}\rho^n e^{-i^*(k_x i h_x + k_y j h_y + k_z k h_z)}, \tag{5.2}$$

where $i^* = \sqrt{-1}$, the nonzero vector $[E_0, \quad H_0]^T$ is the eigenvector of the scheme (4.10), and ρ is the stability factor depending on time whose modulus will determine the stability and dissipation of the scheme under discussion, and $\vec{k} = (k_x, k_y, k_z)$ is the wave number. Substituting the plane wave solution (5.2) into the scheme (4.10), it leads to

$$\begin{bmatrix} \rho - 1 & \epsilon_x i^*(\sin\theta_x)(\rho+1) \\ \mu_x i^*(\sin\theta_x)(\rho+1) & \rho - 1 \end{bmatrix}\begin{bmatrix} E_0 \\ H_0 \end{bmatrix} = 0, \tag{5.3}$$

where $\theta_x = \frac{1}{2}k_x h_x$. The coefficient determinant of the homogeneous algebraic system (5.3) is zero in that the eigenvector $[E_0, \quad H_0]^T$ is nonzero, that is,

$$\rho^2 - 2\frac{1-\mu_x\epsilon_x\sin^2\theta_x}{1+\mu_x\epsilon_x\sin^2\theta_x}\rho + 1 = 0. \tag{5.4}$$

It is easy to verify that the magnitude of the characteristic roots is

$$|\rho| = \frac{1+\mu_x\epsilon_x\sin^2\theta_x}{1+\mu_x\epsilon_x\sin^2\theta_x} = 1.$$

Therefore, the scheme (4.10) is unconditionally stable and non-dissipative. Similarly, we can get the same characteristic roots for the schemes (4.11)–(4.15). It can be concluded that all of the schemes (4.10)–(4.15) are LOD and non-dissipative through each sub-step, and therefore assure the whole unconditionally stable and non-dissipative. Then this leads to the following result.

Theorem 1. *The LOD-MS* (4.10)–(4.15) *is unconditionally stable regardless of the time step τ. The Courant condition is then removed. This also shows that the LOD-MS is non-dissipative.*

To discuss the dispersion of the LOD-MS (4.2)–(4.4), we first analyze the dispersion of their continuous problems (3.9)–(3.11). Because their dispersion is similar, we only discuss it for subproblem (3.9). Substituting a plane wave solution

$$EH_x = e^{i^*(k_x x + k_y y + k_z z - \omega t)} EH_{x0},$$

where EH_{x0} is a nonzero eigenvector, and ω is the frequency, into (3.9), it yields

$$\begin{bmatrix} 0 & -\frac{1}{\epsilon}k_x & -\frac{1}{2}\omega & 0 \\ \frac{1}{\epsilon}k_x & 0 & 0 & \frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 & 0-\frac{1}{\epsilon}k_x & \\ 0 & \frac{1}{2}\omega & -\frac{1}{\epsilon}k_x & 0 \end{bmatrix} EH_{x0} = 0, \tag{5.5}$$

The coefficient determinant must be zero since the eigenvector is nonzero. Thus, we have

$$\left(\frac{1}{4}\omega^2 - v^2 k_x^2\right)^2 = 0, \tag{5.6}$$

where $v = \sqrt{\frac{1}{\epsilon\mu}}$, i.e.,

$$\frac{1}{4}\omega^2 - \frac{1}{\epsilon\mu}k_x^2 = 0. \tag{5.7}$$

In a similar way, we can find that the other two dispersive relations for subproblems (3.10) and (3.11), respectively

$$\frac{1}{4}\omega^2 - \frac{1}{\epsilon\mu}k_y^2 = 0, \quad \frac{1}{4}\omega^2 - \frac{1}{\epsilon\mu}k_z^2 = 0. \tag{5.8}$$

Since the temporal step size for LOD-MS (4.10)–(4.15) is $\frac{1}{2}\tau$, let the stability factor $\rho = e^{\frac{1}{2}i^*\omega\tau}$. Then, substituting ρ into the characteristic equation (5.4) results in

$$(1+\mu_x\epsilon_x\sin^2\theta_x)e^{i^*\omega\tau} - 2(1-\mu_x\epsilon_x\sin^2\theta_x)e^{\frac{1}{2}i^*\omega\tau} + (1+\mu_x\epsilon_x\sin^2\theta_x) = 0. \tag{5.9}$$

Inserting the expressions of $\mu_x, \ \epsilon_x$ and θ_x into (5.9), it brings up

$$v^2 k_x^2 \frac{\sin^2\frac{1}{2}k_x h_x}{\left(\frac{1}{2}k_x h_x\right)^2} = \frac{1}{4}\frac{\tan^2\frac{1}{4}\omega\tau}{\left(\frac{1}{4}\omega\tau\right)^2}\omega^2. \tag{5.10}$$

It is remarked that the dispersive relation (5.10) converges to the theoretical dispersion relation (5.7) provided that τ and h_x both tend to zeros. Similarly, we can see that the numerical dispersive relations for (4.12) and (4.14) are

$$v^2 k_y^2 \frac{\sin^2\frac{1}{2}k_y h_y}{\left(\frac{1}{2}k_y h_y\right)^2} = \frac{1}{4}\frac{\tan^2\frac{1}{4}\omega\tau}{\left(\frac{1}{4}\omega\tau\right)^2}\omega^2, \quad v^2 k_z^2 \frac{\sin^2\frac{1}{2}k_z h_z}{\left(\frac{1}{2}k_z h_z\right)^2} = \frac{1}{4}\frac{\tan^2\frac{1}{4}\omega\tau}{\left(\frac{1}{4}\omega\tau\right)^2}\omega^2, \tag{5.11}$$

which converge to their continuous cases (5.8), respectively, as $\tau, \ h_y, \ h_z$ tend to zeros. The numerical dispersion relations for (4.11), (4.13) and (4.15) are the same as (5.10) and (5.11), respectively.

In what follows, we establish the convergence for the LOD-MS (4.10)–(4.15) using Taylor expansion. To this purpose and for convenience, we omit the subscripts in case they are $i+\frac{1}{2}, \ j+\frac{1}{2}, \ k+\frac{1}{2}$.

Let $j = j+1, \ k = k+1$ in Eqs. (4.10) and (4.11), $i = i+1, \ k = k+1$ in equations (4.12) and (4.13), $i = i+1, \ j = j+1$ in Eqs. (4.14) and (4.15), and add them to (4.10)–(4.11), (4.12)–(4.13) and (4.14)–(4.15), respectively, it yields

$$\begin{cases} \frac{1}{\tau}\left(E_x^* - E_x^n\right) - \frac{1}{2\epsilon}\left[\delta_y H_z^{n+1} + \delta_y H_z^*\right] = 0, \\ \frac{1}{\tau}\left(E_x^{n+1} - E_x^*\right) + \frac{1}{2\epsilon}\left[\delta_z H_y^{n+1} + \delta_z H_y^*\right] = 0, \end{cases} \tag{5.12}$$

$$\begin{cases} \frac{1}{\tau}\left(E_y^* - E_y^n\right) + \frac{1}{2\epsilon}\left[\delta_x H_z^* + \delta_x H_z^n\right] = 0, \\ \frac{1}{\tau}\left(E_y^{n+1} - E_y^*\right) - \frac{1}{2\epsilon}\left[\delta_z H_x^{n+1} + \delta_z H_x^*\right] = 0, \end{cases} \tag{5.13}$$

$$\begin{cases} \frac{1}{\tau}\left(E_z^* - E_z^n\right) - \frac{1}{2\epsilon}\left[\delta_x H_y^* + \delta_x H_y^n\right] = 0, \\ \frac{1}{\tau}\left(E_z^{n+1} - E_z^*\right) + \frac{1}{2\epsilon}\left[\delta_y H_x^* + \delta_y H_x^n\right] = 0, \end{cases} \tag{5.14}$$

$$\begin{cases} \frac{1}{\tau}\left(H_x^* - H_x^n\right) + \frac{1}{2\mu}\left[\delta_y E_z^{n+1} + \delta_y E_z^*\right] = 0, \\ \frac{1}{\tau}\left(H_x^{n+1} - H_x^*\right) - \frac{1}{2\mu}\left[\delta_z E_y^{n+1} + \delta_z E_y^*\right] = 0, \end{cases} \tag{5.15}$$

$$\begin{cases} \frac{1}{\tau}\left(H_y^* - H_y^n\right) - \frac{1}{2\mu}\left[\delta_x E_z^* + \delta_x E_z^n\right] = 0, \\ \frac{1}{\tau}\left(H_y^{n+1} - H_y^*\right) + \frac{1}{2\mu}\left[\delta_z E_x^{n+1} + \delta_z E_x^*\right] = 0, \end{cases} \tag{5.16}$$

$$\begin{cases} \frac{1}{\tau}\left(H_z^* - H_z^n\right) + \frac{1}{2\mu}\left[\delta_x E_y^* + \delta_x E_y^n\right] = 0, \\ \frac{1}{\tau}\left(H_z^{n+1} - H_z^*\right) - \frac{1}{2\mu}\left[\delta_y E_x^* + \delta_y E_x^n\right] = 0. \end{cases} \tag{5.17}$$

From (5.12)–(5.17), adding the first equation to the second one, respectively, it follows that the equivalent scheme for LOD-MS (4.10)–(4.15)

$$\frac{1}{\tau}\left(E_x^{n+1} - E_x^n\right) - \frac{1}{2\epsilon}\left(\delta_y H_z^{n+1} + \delta_y H_z^* - \delta_z H_y^{n+1} - \delta_z H_y^*\right) = 0, \tag{5.18}$$

$$\frac{1}{\tau}\left(E_y^{n+1} - E_y^n\right) + \frac{1}{2\epsilon}\left(\delta_x H_z^* + \delta_x H_z^n - \delta_z H_x^{n+1} - \delta_z H_x^*\right) = 0, \tag{5.19}$$

$$\frac{1}{\tau}\left(E_z^{n+1} - E_z^n\right) - \frac{1}{2\epsilon}\left(\delta_x H_y^* + \delta_x H_y^n - \delta_y H_x^* - \delta_y H_x^n\right) = 0, \tag{5.20}$$

$$\frac{1}{\tau}\left(H_x^{n+1} - H_x^n\right) + \frac{1}{2\mu}\left(\delta_y E_z^{n+1} + \delta_y E_z^* - \delta_z E_y^{n+1} - \delta_z E_y^*\right) = 0, \tag{5.21}$$

$$\frac{1}{\tau}\left(H_y^{n+1} - H_y^*\right) - \frac{1}{2\mu}\left(\delta_x E_z^* + \delta_x E_z^n - \delta_z E_x^{n+1} - \delta_z E_x^*\right) = 0, \tag{5.22}$$

$$\frac{1}{\tau}\left(H_z^{n+1} - H_z^*\right) + \frac{1}{2\mu}\left(\delta_x E_y^* + \delta_x E_y^n - \delta_y E_x^* - \delta_y E_x^n\right) = 0. \tag{5.23}$$

By Taylor's expansion, it figures out that the truncation error for the equivalent scheme (5.18)–(5.23) at the point $\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}}, t^n\right)$ is

$$\begin{aligned} \eta E^n_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\tau\left[\overline{\frac{\partial^2 E_x}{\partial t^2}} - \frac{3}{2\epsilon}\left(\widetilde{\frac{\partial^2 H_z}{\partial t \partial y}} - \widehat{\frac{\partial^2 H_y}{\partial t \partial z}}\right)\right] + \frac{1}{8}\frac{\partial}{\partial t}\left(h_x^2 \overline{\frac{\partial^2 E_x}{\partial x^2}} + h_y^2 \widetilde{\frac{\partial^2 E_x}{\partial y^2}} + h_z^2 \widehat{\frac{\partial^2 E_x}{\partial z^2}}\right) \\ &\quad - \frac{1}{24\epsilon}\left(3h_x^2 \overline{\frac{\partial^3 H_z}{\partial y \partial x^2}} + h_y^2 \widetilde{\frac{\partial^3 H_z}{\partial y^3}} + 3h_z^2 \widehat{\frac{\partial^3 H_z}{\partial y \partial z^2}} - 3h_x^2 \overline{\frac{\partial^3 H_y}{\partial z \partial x^2}} - 3h_y^2 \widetilde{\frac{\partial^3 H_y}{\partial y^2 \partial z}} - h_z^2 \widehat{\frac{\partial^3 H_y}{\partial z^3}}\right), \\ \eta E^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\tau\left[\overline{\frac{\partial^2 E_y}{\partial t^2}} + \frac{1}{2\epsilon}\left(\widetilde{\frac{\partial^2 H_z}{\partial t \partial x}} - 3\widehat{\frac{\partial^2 H_x}{\partial t \partial z}}\right)\right] + \frac{1}{8}\frac{\partial}{\partial t}\left(h_x^2 \overline{\frac{\partial^2 E_y}{\partial x^2}} + h_y^2 \widetilde{\frac{\partial^2 E_y}{\partial y^2}} + h_z^2 \widehat{\frac{\partial^2 E_y}{\partial z^2}}\right) \\ &\quad + \frac{1}{24\epsilon}\left(h_x^2 \overline{\frac{\partial^3 H_z}{\partial x^3}} + 3h_y^2 \widetilde{\frac{\partial^3 H_z}{\partial x \partial y^2}} + 3h_z^2 \widehat{\frac{\partial^3 H_z}{\partial x \partial z^2}} - 3h_x^2 \overline{\frac{\partial^3 H_x}{\partial z \partial x^2}} - 3h_y^2 \widetilde{\frac{\partial^3 H_x}{\partial y \partial z^2}} - h_z^2 \widehat{\frac{\partial^3 H_x}{\partial z^3}}\right), \\ \eta E^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} &= \frac{1}{2}\tau\left[\overline{\frac{\partial^2 E_z}{\partial t^2}} - \frac{1}{\epsilon}\left(\widetilde{\frac{\partial^2 H_y}{\partial t \partial x}} - \widehat{\frac{\partial^2 H_x}{\partial t \partial y}}\right)\right] + \frac{1}{8}\frac{\partial}{\partial t}\left(h_x^2 \overline{\frac{\partial^2 E_z}{\partial x^2}} + h_y^2 \widetilde{\frac{\partial^2 E_z}{\partial y^2}} + h_z^2 \widehat{\frac{\partial^2 E_z}{\partial z^2}}\right) \\ &\quad + \frac{1}{24\epsilon}\left(h_x^2 \overline{\frac{\partial^3 H_y}{\partial x^3}} + 3h_y^2 \widetilde{\frac{\partial^3 H_y}{\partial x \partial y^2}} + 3h_z^2 \widehat{\frac{\partial^3 H_y}{\partial x \partial z^2}} - 3h_x^2 \overline{\frac{\partial^3 H_x}{\partial y \partial x^2}} - h_y^2 \widetilde{\frac{\partial^3 H_x}{\partial y^3}} - 3h_z^2 \widehat{\frac{\partial^3 H_x}{\partial x \partial z^2}}\right), \end{aligned}$$

$$\zeta H^n_{x_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} = \frac{1}{2}\tau\left[\overline{\frac{\partial^2 H_x}{\partial t^2}} + \frac{3}{2\mu}\left(\widetilde{\frac{\partial^2 E_z}{\partial t\partial y}} - \widehat{\frac{\partial^2 E_y}{\partial t\partial z}}\right)\right] + \frac{1}{8}\frac{\partial}{\partial t}\left(h_x^2\overline{\frac{\partial^2 H_x}{\partial x^2}} + h_y^2\widetilde{\frac{\partial^2 H_x}{\partial y^2}} + h_z^2\widehat{\frac{\partial^2 H_x}{\partial z^2}}\right)$$
$$+\frac{1}{24\mu}\left(3h_x^2\overline{\frac{\partial^3 E_z}{\partial y\partial x^2}} + h_y^2\widetilde{\frac{\partial^3 E_z}{\partial y^3}} + 3h_z^2\widehat{\frac{\partial^3 E_z}{\partial y\partial z^2}} - 3h_x^2\overline{\frac{\partial^3 E_y}{\partial z\partial x^2}} - 3h_y^2\widetilde{\frac{\partial^3 E_y}{\partial y^2\partial z}} - h_z^2\widehat{\frac{\partial^3 E_y}{\partial z^3}}\right),$$
$$\zeta H^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} = \frac{1}{2}\tau\left[\overline{\frac{\partial^2 H_y}{\partial t^2}} + \frac{1}{2\mu}\left(\widetilde{\frac{\partial^2 E_z}{\partial t\partial x}} - 3\widehat{\frac{\partial^2 E_x}{\partial t\partial z}}\right)\right] + \frac{1}{8}\frac{\partial}{\partial t}\left(h_x^2\overline{\frac{\partial^2 H_y}{\partial x^2}} + h_y^2\widetilde{\frac{\partial^2 H_y}{\partial y^2}} + h_z^2\widehat{\frac{\partial^2 H_y}{\partial z^2}}\right)$$
$$-\frac{1}{24\mu}\left(h_x^2\overline{\frac{\partial^3 E_z}{\partial x^3}} + 3h_y^2\widetilde{\frac{\partial^3 E_z}{\partial x\partial y^2}} + 3h_z^2\widehat{\frac{\partial^3 E_z}{\partial x\partial z^2}} - 3h_x^2\overline{\frac{\partial^3 E_x}{\partial z\partial x^2}} - 3h_y^2\widetilde{\frac{\partial^3 E_x}{\partial y\partial z^2}} - h_z^2\widehat{\frac{\partial^3 E_x}{\partial z^3}}\right),$$
$$\zeta H^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} = \frac{1}{2}\tau\left[\overline{\frac{\partial^2 H_z}{\partial t^2}} - \frac{1}{\mu}\left(\widetilde{\frac{\partial^2 E_y}{\partial t\partial x}} - \widehat{\frac{\partial^2 E_x}{\partial t\partial y}}\right)\right] + \frac{1}{8}\frac{\partial}{\partial t}\left(h_x^2\overline{\frac{\partial^2 H_z}{\partial x^2}} + h_y^2\widetilde{\frac{\partial^2 H_z}{\partial y^2}} + h_z^2\widehat{\frac{\partial^2 H_z}{\partial z^2}}\right)$$
$$-\frac{1}{24\mu}\left(h_x^2\overline{\frac{\partial^3 E_y}{\partial x^3}} + 3h_y^2\widetilde{\frac{\partial^3 E_y}{\partial x\partial y^2}} + 3h_z^2\widehat{\frac{\partial^3 E_y}{\partial x\partial z^2}} - 3h_x^2\overline{\frac{\partial^3 E_x}{\partial y\partial x^2}} - h_y^2\widetilde{\frac{\partial^3 E_x}{\partial y^3}} - 3h_z^2\widehat{\frac{\partial^3 E_x}{\partial x\partial z^2}}\right),$$

where the notations $\bar{u}$, $\tilde{u}$, and $\hat{u}$ indicate that the function $u(x,y,z,t)$ is calculated at somewhere of the mesh cell $[x_i,x_{i+1}]\times[y_j,y_{j+1}]\times[z_k,z_{k+1}]\times[t^n,t^{n+1}]$.

By summarizing the above analysis, we get the conclusion:

Theorem 2. *The truncation error of the LOD-MS is of first order in temporal direction and of second order in spatial direction. The order in time is one order lower than the methods employed.*

Next, we discuss the energy conservation property of the LOD-MS (4.10)–(4.15).

Theorem 3. *Suppose that the initial values $[\mathbf{E}(x,y,z,0),\mathbf{H}(x,y,z,0)]^T$ are symmetric about the spatial variables x, y and z, and the spatial domain under consideration is cubic or ball whose center is the origin, and the mesh step sizes are uniform, i.e., $h_x=h_y=h_z$. For any integer $n\geqslant 0$, set $\mathbf{E}^n_{i,j,k}=\left(E^n_{x_{i+\frac{1}{2},j,k}},E^n_{y_{i,j+\frac{1}{2},k}},E^n_{z_{i,j,k+\frac{1}{2}}}\right)$ and $\mathbf{H}^n_{i,j,k}=\left(H^n_{x_{i+\frac{1}{2},j,k}},H^n_{y_{i,j+\frac{1}{2},k}},H^n_{z_{i,j,k+\frac{1}{2}}}\right)$ be the solution of the LOD-MS (4.2)–(4.4),* the LOD-MS (4.10)–(4.15) conserve the discrete version of the energy conservation laws (2.8) and (2.9), that is,

$$\epsilon\|\mathbf{E}^{n+1}\|^2_{\frac{1}{2}} + \mu\|\mathbf{H}^{n+1}\|^2_{\frac{1}{2}} = \epsilon\|\mathbf{E}^{n}\|^2_{\frac{1}{2}} + \mu\|\mathbf{H}^{n}\|^2_{\frac{1}{2}} = \cdots = \epsilon\|\mathbf{E}^{0}\|^2_{\frac{1}{2}} + \mu\|\mathbf{H}^{0}\|^2_{\frac{1}{2}}, \tag{5.24}$$
$$\epsilon\|\delta_t\mathbf{E}^{n+1}\|^2_{\frac{1}{2}} + \mu\|\delta_t\mathbf{H}^{n+1}\|^2_{\frac{1}{2}} = \epsilon\|\delta_t\mathbf{E}^{n}\|^2_{\frac{1}{2}} + \mu\|\delta_t\mathbf{H}^{n}\|^2_{\frac{1}{2}} = \epsilon\|\delta_t\mathbf{E}^{0}\|^2_{\frac{1}{2}} + \mu\|\delta_t\mathbf{H}^{0}\|^2_{\frac{1}{2}}, \tag{5.25}$$

where

$$\|\mathbf{E}^n\|^2_{\frac{1}{2}} = h_xh_yh_z\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\left(E^n_{x_{i+\frac{1}{2},j,k}}\right)^2 + \left(E^n_{y_{i,j+\frac{1}{2},k}}\right)^2 + \left(E^n_{z_{i,j,k+\frac{1}{2}}}\right)^2\right),$$
$$\|\mathbf{H}^n\|^2_{\frac{1}{2}} = h_xh_yh_z\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\left(H^n_{x_{i+\frac{1}{2},j,k}}\right)^2 + \left(H^n_{y_{i,j+\frac{1}{2},k}}\right)^2 + \left(H^n_{z_{i,j,k+\frac{1}{2}}}\right)^2\right).$$

Proof. Multiplying both sides of (4.2)–(4.4) with $Z^{(1)^{n+\frac{1}{2}*}}_{i+\frac{1}{2},j,k}$, $Z^{(2)^{n+\frac{1}{2}}}_{i,j+\frac{1}{2},k}$, $Z^{(3)^{n+\frac{1}{2}}}_{i,j,k+\frac{1}{2}}$, here $Z^{(1)^{n+\frac{1}{2}*}}_{i+\frac{1}{2},j,k}=\Big[\epsilon E^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j,k}},\ \mu E^{n+\frac{1}{2}*}_{z_{i+\frac{1}{2},j,k}},\epsilon H^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j,k}},$ $\mu H^{n+\frac{1}{2}*}_{z_{i+\frac{1}{2},j,k}}\Big]^T$, $Z^{(2)^{n+\frac{1}{2}}}_{i,j+\frac{1}{2},k}=\left[\epsilon E^{n+\frac{1}{2}*}_{x_{i,j+\frac{1}{2},k}},\mu E^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2},k}},\epsilon H^{n+\frac{1}{2}*}_{x_{i,j+\frac{1}{2},k}},\mu H^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2},k}}\right]^T$, $Z^{(3)^{n+\frac{1}{2}}}_{i,j,k+\frac{1}{2}}=\left[\epsilon E^{n+\frac{1}{2}}_{x_{i,j,k+\frac{1}{2}}},\mu E^{n+\frac{1}{2}}_{y_{i,j,k+\frac{1}{2}}},\epsilon H^{n+\frac{1}{2}}_{x_{i,j,k+\frac{1}{2}}},\mu H^{n+\frac{1}{2}}_{y_{i,j,k+\frac{1}{2}}}\right]^T$, respectively, it yields

Table 1
Spatial accuracy with $\tau=0.01$.

h	$\|e_{E_z}\|_2$	$Order_2$	$\|e_{E_z}\|_\infty$	$Order_\infty$	$\|e_{H_y}\|_2$	$Order_2$	$\|e_{H_y}\|_\infty$	$Order_\infty$
$\frac{2\pi}{50}$	5.856E−3	–	9.312E−3	–	4.140E−3	–	6.585E−3	–
$\frac{2\pi}{100}$	1.033E−3	2.50	2.326E−3	2.00	7.310E−4	2.50	1.644E−3	2.00
$\frac{2\pi}{200}$	1.826E−4	2.50	5.813E−4	2.00	1.291E−4	2.50	4.110E−4	2.00
$\frac{2\pi}{400}$	3.223E−5	2.50	1.451E−4	2.00	2.279E−5	2.50	1.026E−4	2.00

$$\begin{cases} \frac{\epsilon}{2\tau}\left(\left(E^{*}_{y_{i+\frac{1}{2},j,k}}\right)^2-\left(E^{n}_{y_{i+\frac{1}{2},j,k}}\right)^2\right)=-E^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j,k}}\delta_x H^{n+\frac{1}{2}*}_{z_{i,j,k}},\\ \frac{\mu}{2\tau}\left(\left(H^{*}_{z_{i+\frac{1}{2},j,k}}\right)^2-\left(H^{n}_{z_{i+\frac{1}{2},j,k}}\right)^2\right)=-H^{n+\frac{1}{2}*}_{z_{i+\frac{1}{2},j,k}}\delta_x E^{n+\frac{1}{2}*}_{y_{i,j,k}},\\ \frac{\epsilon}{2\tau}\left(\left(E^{*}_{z_{i+\frac{1}{2},j,k}}\right)^2-\left(E^{n}_{z_{i+\frac{1}{2},j,k}}\right)^2\right)=E^{n+\frac{1}{2}*}_{z_{i+\frac{1}{2},j,k}}\delta_x H^{n+\frac{1}{2}*}_{y_{i,j,k}},\\ \frac{\mu}{2\tau}\left(\left(H^{*}_{y_{i+\frac{1}{2},j,k}}\right)^2-\left(H^{n}_{y_{i+\frac{1}{2},j,k}}\right)^2\right)=H^{n+\frac{1}{2}*}_{y_{i+\frac{1}{2},j,k}}\delta_x E^{n+\frac{1}{2}*}_{z_{i,j,k}}, \end{cases}$$

$$\begin{cases} \frac{\epsilon}{2\tau}\left(\left(E^{*}_{x_{i,j+\frac{1}{2},k}}\right)^2-\left(E^{n}_{x_{i,j+\frac{1}{2},k}}\right)^2\right)=E^{n+\frac{1}{2}*}_{x_{i,j+\frac{1}{2},k}}\delta_y H^{n+\frac{1}{2}}_{z_{i,j,k}},\\ \frac{\mu}{2\tau}\left(\left(H^{n+1}_{z_{i,j+\frac{1}{2},k}}\right)^2-\left(H^{*}_{z_{i,j+\frac{1}{2},k}}\right)^2\right)=H^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2},k}}\delta_y E^{n+\frac{1}{2}*}_{x_{i,j,k}},\\ \frac{\epsilon}{2\tau}\left(\left(E^{n+1}_{z_{i,j+\frac{1}{2},k}}\right)^2-\left(E^{*}_{z_{i,j+\frac{1}{2},k}}\right)^2\right)=-E^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2},k}}\delta_y H^{n+\frac{1}{2}*}_{x_{i,j,k}},\\ \frac{\mu}{2\tau}\left(\left(H^{*}_{x_{i,j+\frac{1}{2},k}}\right)^2-\left(H^{n}_{x_{i,j+\frac{1}{2},k}}\right)^2\right)=-H^{n+\frac{1}{2}*}_{x_{i,j+\frac{1}{2},k}}\delta_y E^{n+\frac{1}{2}}_{z_{i,j,k}}, \end{cases}$$

$$\begin{cases} \frac{\epsilon}{2\tau}\left(\left(E^{n+1}_{x_{i,j,k+\frac{1}{2}}}\right)^2-\left(E^{*}_{x_{i,j,k+\frac{1}{2}}}\right)^2\right)=-E^{n+\frac{1}{2}}_{x_{i,j,k+\frac{1}{2}}}\delta_z H^{n+\frac{1}{2}}_{z_{i,j,k}},\\ \frac{\mu}{2\tau}\left(\left(H^{n+1}_{y_{i,j,k+\frac{1}{2}}}\right)^2-(H^{*}_{y_{i,j,k+\frac{1}{2}}})^2\right)=-H^{n+\frac{1}{2}}_{y_{i,j,k+\frac{1}{2}}}\delta_z E^{n+\frac{1}{2}}_{x_{i,j,k}},\\ \frac{\epsilon}{2\tau}\left(\left(E^{n+1}_{y_{i,j,k+\frac{1}{2}}}\right)^2-\left(E^{*}_{y_{i,j,k+\frac{1}{2}}}\right)^2\right)=E^{n+\frac{1}{2}}_{y_{i,j,k+\frac{1}{2}}}\delta_z H^{n+\frac{1}{2}}_{x_{i,j,k}},\\ \frac{\mu}{2\tau}\left(\left(H^{n+1}_{x_{i,j,k+\frac{1}{2}}}\right)^2-\left(H^{*}_{x_{i,j,k+\frac{1}{2}}}\right)^2\right)=H^{n+\frac{1}{2}}_{x_{i,j,k+\frac{1}{2}}}\delta_z E^{n+\frac{1}{2}}_{y_{i,j,k}}. \end{cases}$$

Summing all terms in the above equations over all spatial indices i, j, k, and adding them together, note that the assumption and Green formula, the right hand side is offset. All U^*-valued including terms on the left side are vanished. The remainder is just the first energy conservation law (5.24).

To prove the second discrete energy conservation law (5.25), we introduce two notations: u^{n+1*} is the intermediate value u^* at the $(n+1)$th level, and $\delta_t u^{n+\frac{1}{2}*} = \frac{u^{n+1*}-u^{n*}}{\tau}$. Acting the forward temporal difference quotient operator δ_t on each term of (4.2)–(4.4), then multiplying the resulting equalities with $\delta_t Z^{(1)n+\frac{1}{2}*}_{i+\frac{1}{2},j,k}$, $\delta_t Z^{(2)n+\frac{1}{2}}_{i,j+\frac{1}{2},k}$, $\delta_t Z^{(3)n+\frac{1}{2}}_{i,j,k+\frac{1}{2}}$ on both sides, respectively, following

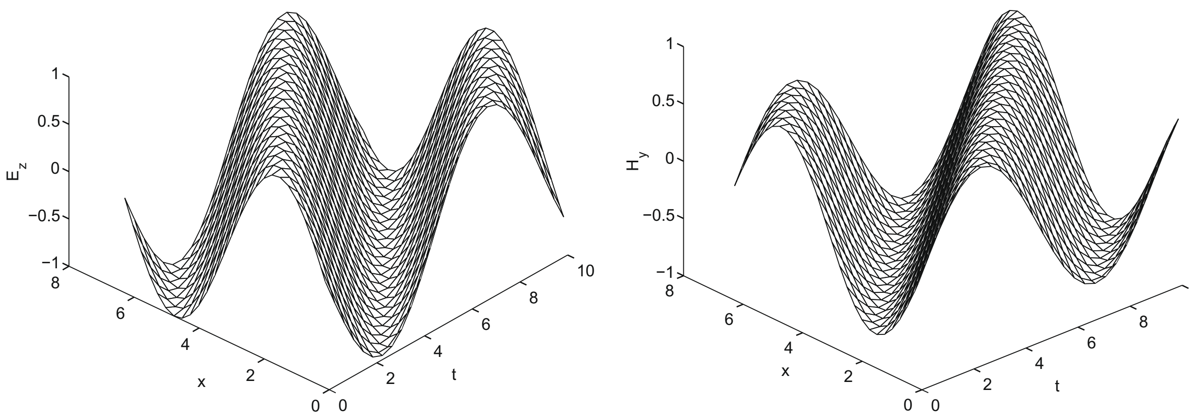


Fig. 1. The profiles of E_z and H_y.

the proof of the discrete conservation law (5.24), we get the second discrete energy conservation law (5.25). In the argument, we have employed the commutability between temporal difference quotient operator and spatial ones. The proof is finished. □

6. Numerical results

This section will provide numerical experiments to test the new derived LOD-MS (4.10)–(4.15). We will show some numerical results for one-dimensional, two-dimensional and three-dimensional Maxwell's equations with constant electric permittivity and magnetic permeability. The main work focuses on the convergence and conservation laws, and stability as well.

Suppose that $u(x_i, y_j, z_k, t^n)$ and $u^n_{i,j,k}$ are the exact solution of the differential equations and the approximation of the LOD-MS at node (x_i, y_j, z_k, t^n), respectively. As usual, we consider the numerical error in L_∞ and L_2 norm, respectively, defined by

$$\|e_u\|_\infty = \max_{i,j,k} |u(x_i, y_j, z_k, t^n) - u^n_{i,j,k}|,$$

$$\|e_u\|_2 = h_x h_y h_z \left(\sum_{i,j,k} (u(x_i, y_j, z_k, t^n) - u^n_{i,j,k})^2 \right)^{\frac{1}{2}}.$$

6.1. One-dimensional test problem

To check the numerical accuracy of central box scheme (3.20), we first consider a simple example of one-dimensional transverse magnetic wave

$$\begin{cases} \frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \frac{\partial H_y}{\partial x}, \\ \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \frac{\partial E_z}{\partial x}, \end{cases} \tag{6.1}$$

by prescribing initial conditions

$$E_z(x, 0) = \sin(x), \quad H_y(x, 0) = -\sqrt{\frac{\epsilon}{\mu}} \sin(x),$$

and perfectly electric conducting boundary conditions. In this case, problem (6.1) has an exact solution

$$E_z(x, t) = \sin\left(x - \sqrt{\frac{1}{\mu\epsilon}} t\right), \quad H_y(x, t) = -\sqrt{\frac{\epsilon}{\mu}} \sin\left(x - \sqrt{\frac{1}{\mu\epsilon}} t\right). \tag{6.2}$$

In such a case, both energies I (2.8) and II (2.9) are kept exactly provided that the problem is simulated by the multisymplectic central box scheme (3.20). We choose $\mu = \epsilon = 1$. In the experiment, the problem is discretized on the spatial–temporal domain $[0, 2\pi] \times [0, 10]$ with time step size $\tau = 0.01$ and with different spatial step lengths. The numerical errors between

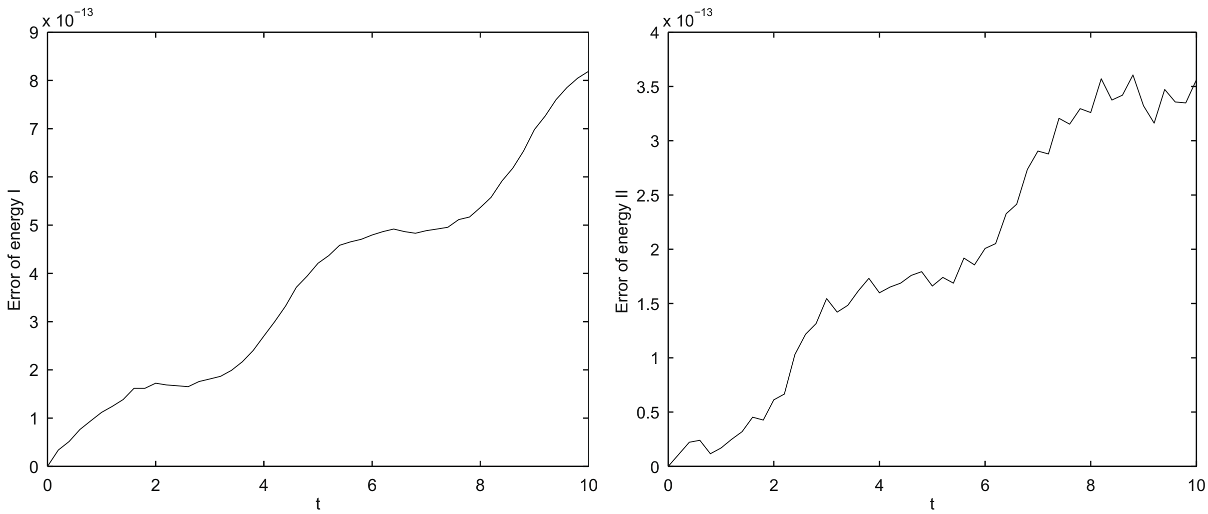
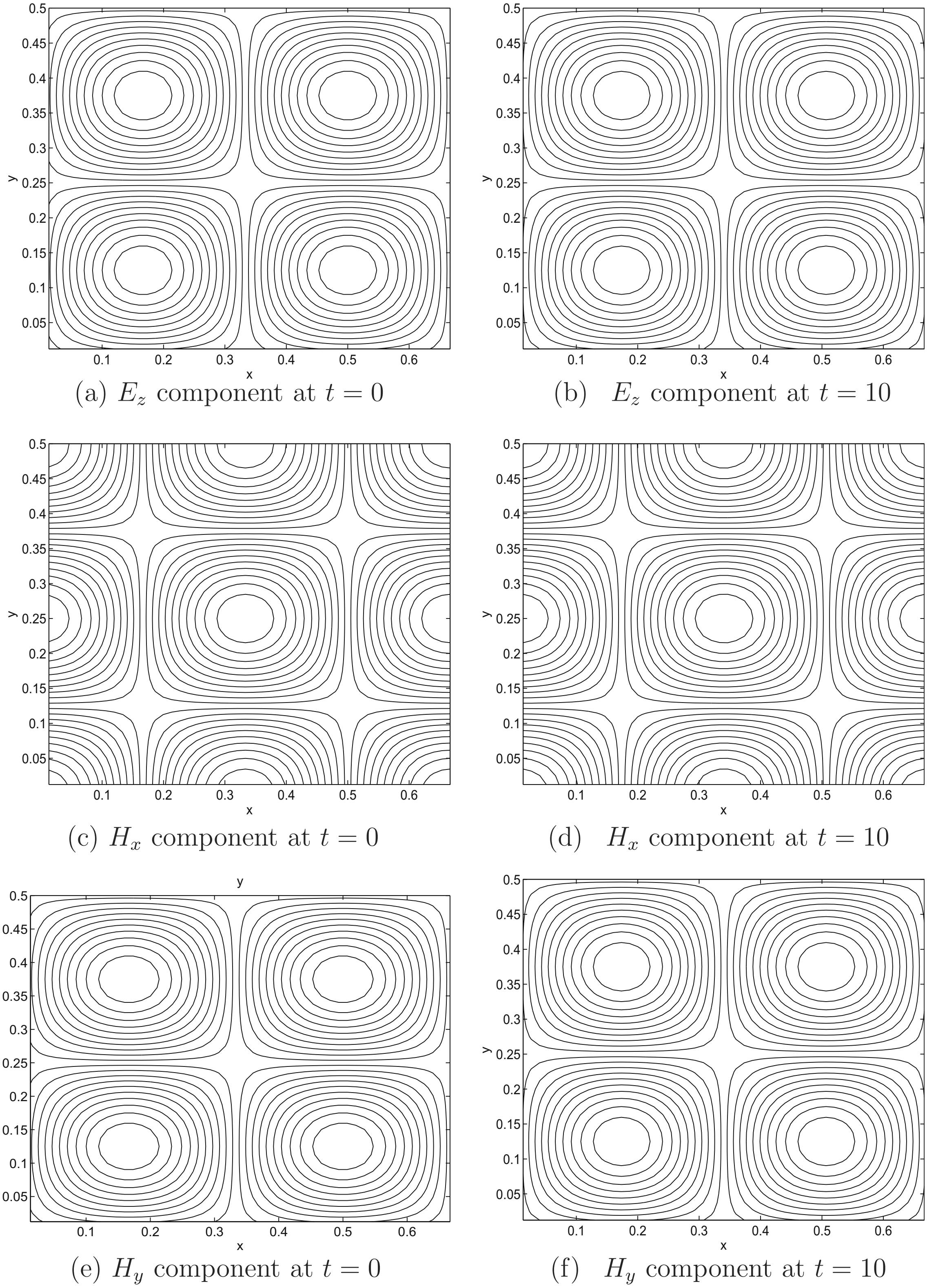


Fig. 2. Error of energy I and II.



(a) E_z component at $t = 0$

(b) E_z component at $t = 10$

(c) H_x component at $t = 0$

(d) H_x component at $t = 10$

(e) H_y component at $t = 0$

(f) H_y component at $t = 10$

Fig. 3. 20 Contours of the 2-D TM wave at different times.

exact solution and numerical solution in L_∞ and L_2 norms are enumerated in Table 1. The profiles of the electric component E_z and magnetic component H_y are shown in Fig. 1. The errors of the conserved quantities (2.8) and (2.9) are pictured in

Fig. 2. From the table and figures, we know that the central box scheme (3.20) is of second order convergence rate. It conserves the energies I and II indeed.

6.2. Two-dimensional problem

In the following experiment, we concert the transverse magnetic (TM) polarization case inside a perfectly electric conducting domain. The model reads

$$\begin{cases} \frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\frac{\partial E_z}{\partial y}, \\ \frac{\partial H_y}{\partial t} = \frac{1}{\mu}\frac{\partial E_z}{\partial x}, \\ \frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right). \end{cases} \tag{6.3}$$

The spatial domain concerted is a rectangle $[0, \frac{2}{3}] \times [0, \frac{1}{2}]$. The preset initial conditions are as follows:

$$\begin{aligned} &E_z(x, y, 0) = \sin(3\pi x)\sin(4\pi y), \\ &H_x(x, y, 0) = -0.8\cos(3\pi x)\cos(4\pi y), \\ &H_y(x, y, 0) = -0.6\sin(3\pi x)\sin(4\pi y). \end{aligned} \tag{6.4}$$

The parameters are normalized to $\mu = \epsilon = 1$. For simplicity, we discuss perfectly electric conducting boundary conditions. We apply LOD-MS (4.18) and (4.19) to solve the problem until $t = 10$. The spatial–temporal domain is divided by step sizes

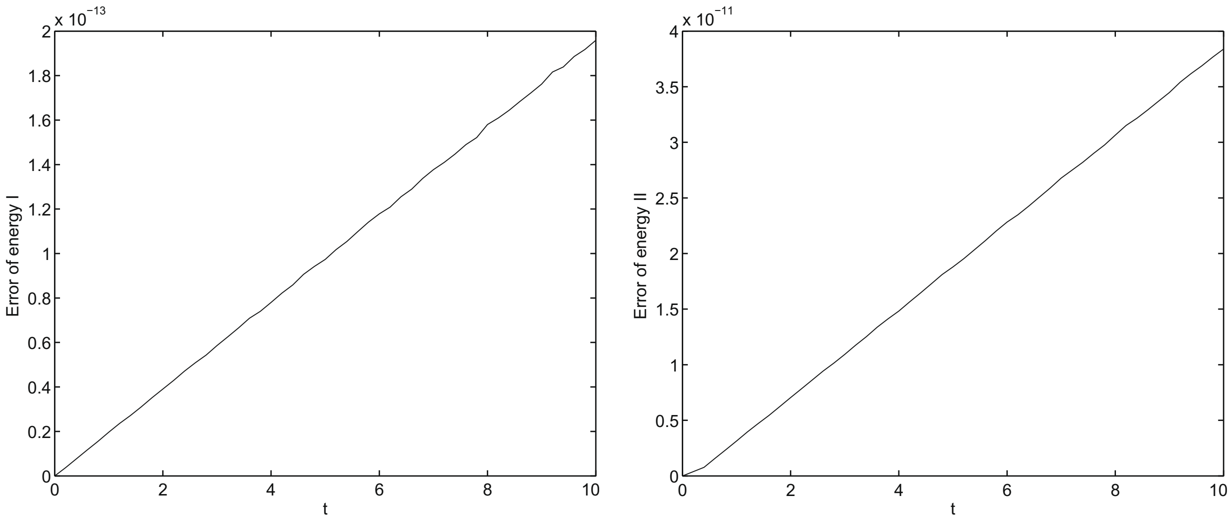


Fig. 4. Error of energy I and II.

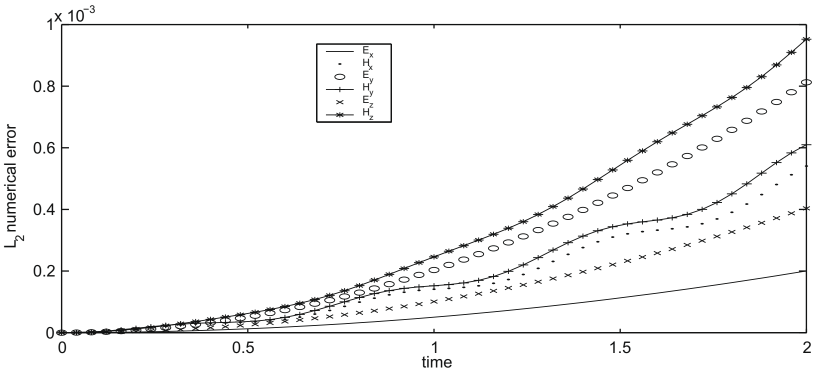


Fig. 5. Error between numerical solution and exact solution in L_2 norm.

$h_x = \frac{2}{300},\ h_y = \frac{1}{160}$ and $\tau = 0.01$. The numerical results with 20 contours for all electric and magnetic components are described in Fig. 3. The two conserved quantities (2.8) and (2.9) are presented in Fig. 4.

Looking at Figs. 3 and 4, it can be observed that the numerical solutions are consistent to the theoretical results of Theorems 2 and 3. The contours are evenly distributed in the x–y plane for all components. The two conserved quantities are preserved actually whose errors are within the roundoff error.

By the way, we had considered to simulate the TE problem (6.3) by the multisymplectic central box scheme (4.17) to compare its efficiency with that of LOD-MS. However, we failed to code it because it need to solve a large scale of algebraic system even if the mesh division is very coarse owning to the limitation of our PC.

6.3. Three-dimensional test

To verified the correctness of the LOD-MS for 3-D Maxwell's equations, we choose the following exactly periodic solution of Maxwell's equations (2.1) and (2.2) over the unit cube with $\mu = \epsilon = 1$:

$$
\begin{aligned}
&E_x = \cos\left(2\pi(x+y+z) - 2\sqrt{3}\pi t\right), \quad H_x = \sqrt{3}E_x,\\
&E_y = -2E_x, \quad H_y = 0,\\
&E_z = E_x, \quad H_z = -\sqrt{3}E_x.
\end{aligned} \tag{6.5}
$$

From the expression and the cubic spatial domain, we can conclude that Theorem 3 is correct provided that the spatial steps $h_x = h_y = h_z$. The expression (6.5) of the solution implies that the waves propagate along the main diagonal of the computational domain.

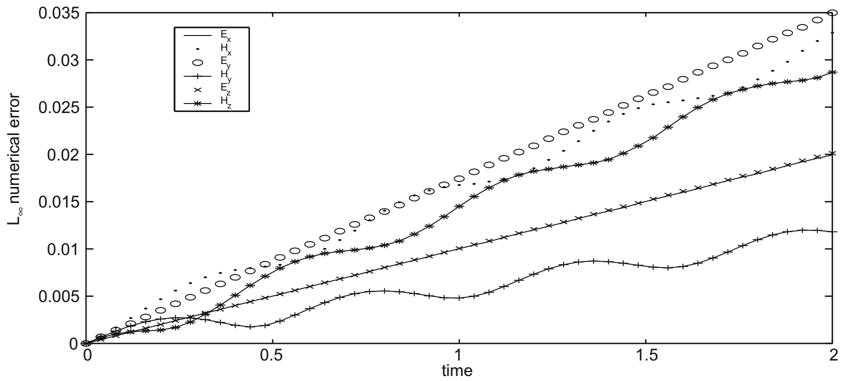


Fig. 6. Error between numerical solution and exact solution in L_∞ norm.

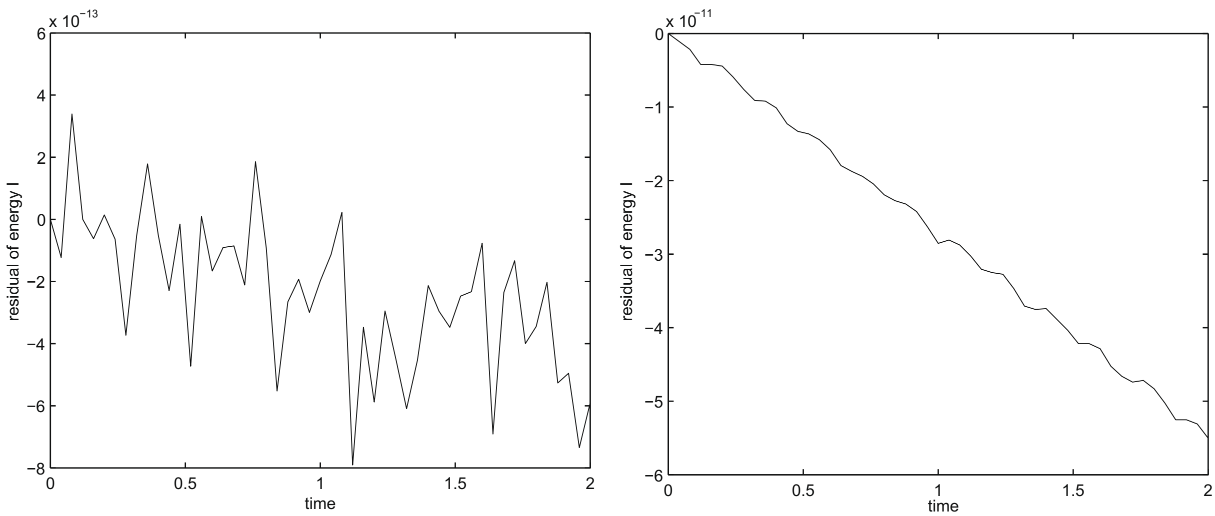


Fig. 7. Residuals of energy I and II with $\tau = \frac{2}{4000}$.

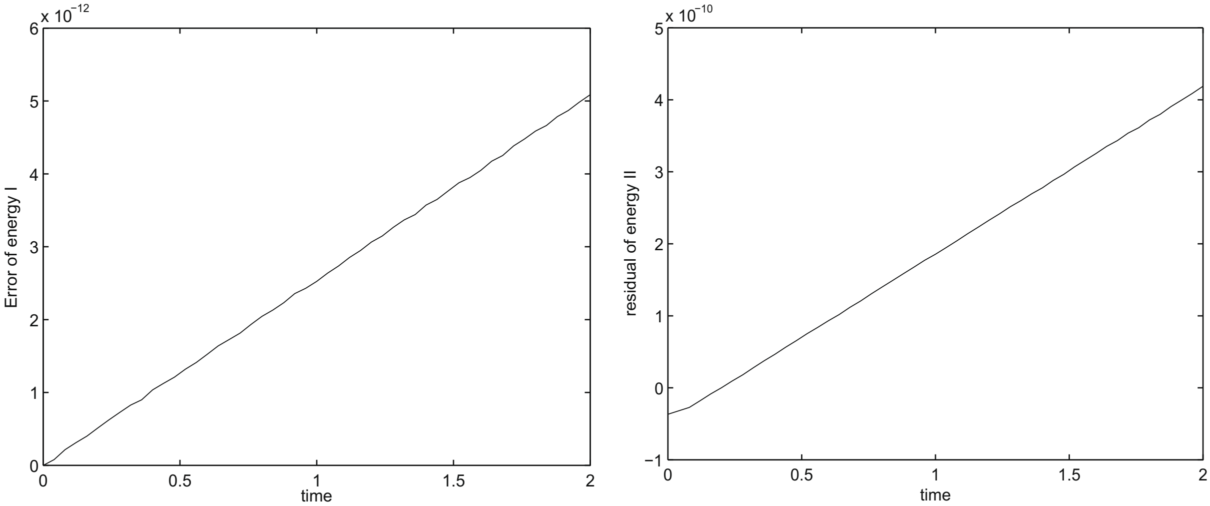


Fig. 8. Residuals of energy I and II with $\tau = \frac{2}{8000}$.

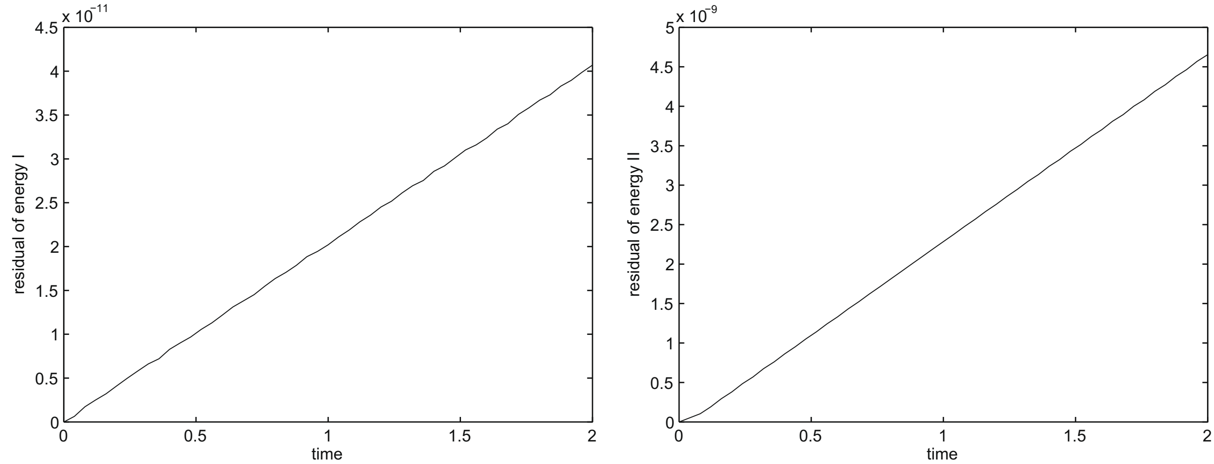


Fig. 9. Residuals of energy I and II with $\tau = \frac{2}{16000}$.

Our numerical schemes (4.10)–(4.15) approximate the exact solution (6.5) over the spatial–temporal domain $[0,1]^3 \times [0,2]$. For the spatial step size, we use $h_x = h_y = h_z = \frac{1}{64}$, and for the temporal step length, we use $\tau = \frac{1}{8000}$. Figs. 5 and 6 show the numerical errors between numerical solution and exact solution in the sense of averaged and maximum norm, respectively, for all components. And Figs. 7–9 provide the residuals of energy I and II with different temporal step sizes $\tau = \frac{2}{4000}$, $\tau = \frac{2}{8000}$ and $\frac{2}{16000}$, respectively. From these figures, we can observed that the numerical solution approximates the exact solution very well. Both energy I and energy II are exactly preserved within the machine precision. As the temporal mesh refined, the residuals of energy grow. The reason mostly lies in the refinement of mesh leading to the increment of computational cost. Thus, the roundoff error is enhanced. Moreover, the residual for the second energy is two orders of magnitude larger than that of the first one, which is caused by the division of τ at every time level in the second energy expression.

7. Conclusions and remarks

We have developed a new type of approach to devise multisymplectic integrators for multi-dimensional HPDEs which have not discovered in the existing literatures. The technique is successfully applied to 1-D, 2-D and 3-D Maxwell's equations. We split the original multisymplectic HPDEs into several LOD HPDEs. The LOD multisymplectic HPDEs are discretized by a pair of SRK methods which lead to directional multisymplecticity, separately, and one is employed as the initial values of the next one. The directional multisymplectic conservation laws are all satisfied when they are applied to Maxwell's equations. Moreover, it is paid for the total symplecticity is preserved indeed. By theoretical and numerical analysis, the LOD-MS maintains the quadratic energy exactly in some cases. However, the convergence rate of the LOD-MS is generally lower than that of the original methods employed in temporal direction, which is paid for the superiors.

The boundary conditions we consider in the paper are perfectly electric conductor, namely, homogeneous boundary conditions. In fact, the LOD-MS method can be generalized to other boundary conditions, such as periodic boundary condition. However, it is difficult using it to discontinuous boundary condition. We will discuss its application to problems with source term and unbounded domain which will be combined with artificial boundary methods [33]. We will investigate other kind of multisymplectic integrators to design LOD-MS for multi-dimensional HPDEs, such as Runge–Kutta–Nyström method, partitioned Runge–Kutta method, etc., and their conservation property. Certainly, it is also very important to widen the application to other context, such as, multi-dimensional Schrödinger equation, Klein–Gordon–Schrödinger equations, etc.

Acknowledgments

The authors would like to express their appreciation to the referees for their useful comments and the editors. Linghua Kong is supported by the Natural Science Foundation of China (No. 10901074), the Provincial Natural Science Foundation of Jiangxi (No. 2008GQS0054), the Foundation of Department of Education Jiangxi Province (No. GJJ09147), the Foundation of Jiangxi Normal University (Nos. 2057, 2390), State Key Laboratory of Scientific and Engineering Computing, CAS, and the Provincial Natural Science Foundation of Anhui (No. 090416227).

Jialin Hong is supported by the Director Innovation Foundation of ICMSEC and AMSS, the Foundation of CAS, the NNSFC (Nos. 19971089, 10371128 and 60771054) and the Special Funds for Major State Basic Research Projects of China 2005CB321701.

References

[1] K. Feng, On difference schemes and symplectic geometry, in: K. Feng (Ed.), Proc. of the 1984 Beijing Symp. on Diff. Geometry and Diff. Equations, Science Press, Beijing, 1985, pp. 42–58.
[2] E. Hairer, C. Lubich, G. Wanner, Geometric Numerical Integration Structure-Preserving Algorithms for Ordinary Differential Equations, second ed., Springer-Verlag, Berlin, 2006.
[3] R. McLachlan, G. Quispel, Splitting methods, Acta Numer. (2002) 341–434.
[4] S. Reich, Multi-symplectic Runge–Kutta collocation methods for Hamiltonian wave equation, J. Comput. Phys. 157 (2000) 473–499.
[5] T.J. Bridges, S. Reich, Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity, Phys. Lett. A 284 (2001) 184–193.
[6] B.N. Ryland, B.I. Mclachlan, J. Frank, On multisymplecticity of partitioned Runge–Kutta and splitting methods, Int. J. Math. Comput. 84 (2007) 847–869.
[7] J. Hong, L.H. Kong, Novel multisymplectic integrators for nonlinear fourth-order Schrödinger equation with trapped term, Commun. Comput. Phys. 7 (2010) 613–630.
[8] J.L. Hong, C. Li, Multi-symplectic Runge–Kutta methods for nonlinear Dirac equations, J. Comput. Phys. 211 (2006) 448–472.
[9] J.L. Hong, S.S. Jiang, C. Li, H.Y. Liu, Explicit multisymplectic methods for Hamiltonian wave equations, Commun. Comput. Phys. 2 (2007) 662–683.
[10] A.L. Islas, C.M. Schober, On the preservation of phase space structure under multisymplectic discretization, J. Comput. Phys. 197 (2004) 585–609.
[11] J.L. Hong, Y. Liu, Hans Munthe-Kaas, Antonella Zanna, Globally conservative properties and error estimation of a multi-symplectic scheme for Schrödinger equations with variable coefficients, Appl. Numer. Math. 56 (2006) 814–843.
[12] J.L. Hong, X.Y. Liu, C. Li, Multi-symplectic Runge–Kutta–Nyström methods for nonlinear Schrödinger equations with variable coefficients, J. Comput. Phys. 226 (2007) 1968–1984.
[13] C.M. Schober, T.H. Whodarczyk, Dispersive properties of multisymplectic integrators, J. Comput. Phys. 227 (2008) 5090–5104.
[14] L. Wang, Multisymplectic Preissman scheme and its application, J. Jiangxi Normal Univ. 33 (2009) 42–46.
[16] J.X. Cai, Y.S. Wang, B. Wang, B. Jiang, New multisympletic self-adjoint scheme and its composition scheme for the time-domain Maxwell's equations, J. Math. Phys. 47 (2006) 123508.
[17] H.L. Su, M.Z. Qin, R. Scherer, A multisymplectic geometry and a multisymplectic scheme for Maxwell's equations, Int. J. Pure Appl. Math. 34 (2007) 1–17.
[18] J. Douglas Jr., H. Rachford, On the numerical solution of heat conduction problems in two and three space variables, Trans. Am. Math. Soc. 82 (1960) 421–439.
[19] D. Peaceman, H. Rachford, The numerical solution of parabolic and elliptic equations, J. Soc. Indust. Appl. Math. 3 (1955) 28–41.
[20] J. Douglas Jr., H.H. Rachford Jr., On the numerical solution of heat conduction problems in two and three space variables, Trans. Am. Math. Soc. 82 (1956) 421–439.
[21] J. Douglas Jr., S. Kim, Improved accuracy for locally one-dimensional methods for parabolic equations, Math. Mod. Meth. Appl. Sci. 11 (2001) 1563–1579.
[22] K.S. Yee, Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media, IEEE Trans. Antennas Propagat. 14 (1966) 302–307.
[23] R. Holland, Implicit three-dimensional finite difference of Maxwell's equations, IEEE Trans. Nucl. Sci. 31 (1984) 1322–1326.
[24] W.B. Chen, X.J. Li, D. Liang, Energy-conserved splitting FDTD methods for Maxwell's equations, Numer. Math. 108 (2008) 445–485.
[25] L.P. Gao, B. Zhang, D. Liang, The splitting finite-difference time-domain methods for Maxwell's equations in two dimensions, J. Comput. Appl. Math. 205 (2007) 207–230.
[26] Z.Q. Xie, C.H. Chan, B. Zhang, An explicit fourth-order staggered finite-difference time-domain method for Maxwell's equations, J. Comput. Appl. Math. 147 (2002) 75–98.
[27] F. Zheng, Z. Chen, J. Zhang, Toward the development of a three-dimensional unconditionally stable finite-difference time-domain method, IEEE Trans. Microwave Theor. Tech. 48 (2000) 1550–1558.
[28] W. Sha, Z.X. Huang, X.L. Wu, M.S. Chen, Application of the symplectic finite-difference time-domain scheme to electromagnetic simulation, J. Comput. Phys. 225 (2007) 33–50.
[29] J. Lee, B. Fornbergb, A split step approach for the 3-D Maxwell's equations, J. Comput. Appl. Math. 158 (2003) 485–505.

[30] J. Lee, B. Fornbergb, Some unconditionally stable time stepping methods for the 3D Maxwell's equations, J. Comput. Appl. Math. 166 (2004) 497–523.
[31] D.B. Ge, Y.B. Yan, Finite-Difference Time-Domain Method for Electromagnetic Wave, Xidian Univer. Press, Xi'an, 2005.
[32] Z.L. Xu, J.S. He, H.D. Han, Semi-implicit operator splitting Padé method for higher-order nonlinear Schrödinger equations, Appl. Math. Comput. 179 (2006) 596–605.
[33] H.D. Han, X.N. Wu, Artificial Boundary Method, Tsinghua University Press, Beijing, 2009 (in Chinese).